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Nonlinear pricing, market coverage, and competition

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This paper considers a nonlinear pricing framework with both horizontally and vertically differentiated products. By endogenizing the set of consumers served in the market, we are able to study how increased competition affects nonlinear pricing, in particular the market coverage and quality distortions. We characterize the symmetric equilibrium menu of price–quality offers under different market structures. When the market structure moves from monopoly to duopoly, we show that more types of consumers are served and quality distortions decrease. As the market structure becomes more competitive, the effect of increased competition exhibits some non-monotonic features: when the initial competition is not too weak, a further increase in the number of firms leads to more types of consumers being covered and a reduction in quality distortions; when the initial competition is weak, an increase in the number of firms leads to fewer types of consumers being covered, though the effect on quality distortions is not uniform.

Keywords. Nonlinear pricing, product differentiation, market coverage, quality distortions.

JEL classification. D40, D82, L10.

1. Introduction

Since the work of Mussa and Rosen (1978) and Maskin and Riley (1984) on monopolistic nonlinear pricing, there has been a growing literature on nonlinear pricing in competitive settings (see, for example, Spulber 1989, Champsaur and Rochet 1989, Wilson 1993, Gilbert and Matutes 1993, Stole 1995, Verboven 1999, Villas-Boas and Schmidt-Mohr 1999, Armstrong and Vickers 2001, 2006, Rochet and Stole 1997, 2002, and Ellison 2005). However, much remains to be done to understand how increased competition affects

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firms' nonlinear pricing strategies. In this paper, we focus on the effects of increased (horizontal) competition on the (vertical) market coverage and quality distortions. In doing so our paper is most closely related to Rochet and Stole (1997, 2002), to which it is complementary.

Rochet and Stole (1997) study duopoly nonlinear pricing in a standard Hotelling model in which the horizontal types of consumers are distributed uniformly over $[0, \Delta]$ and the vertical types of consumers are distributed uniformly over $[\underline{\theta}, \overline{\theta}]$, where $\underline{\theta}/\overline{\theta}$ is larger than some value approximately equal to 0.76. Their main results are as follows. When the degree of horizontal differentiation is sufficiently large ($\Delta \geqslant \Delta_m$) so that each firm is in effect a local monopoly, the equilibrium exhibits perfect sorting, with quality distortions for all types but the top ($\overline{\theta}$) and the bottom ($\underline{\theta}$). When the degree of horizontal differentiation is sufficiently low ($\Delta \leq \Delta_c$), the market is fully covered on both vertical and horizontal dimensions, each firm offers a cost-plus-fee pricing schedule, and quality provision is fully efficient for all the types. Finally, when $\Delta_m > \Delta > \Delta_c$, competition in nonlinear schedules yield a mixed regime consisting of both the local monopoly region and the competitive region. The equilibrium exhibits perfect sorting with quality distortions for all but $\overline{\theta}$ and $\underline{\theta}$.

The analysis in Rochet and Stole (2002) is more general, as it covers both monopoly and duopoly cases, and allows for general distributions for horizontal types of consumers (though vertical types are still assumed to be uniformly distributed). These horizontal types are interpreted as outside opportunity costs, which give rise to random participation by consumers. By taking random participation into account, they show that in the monopoly case there is either bunching or no quality distortion at the bottom. In the duopoly case, they show that under full market coverage, quality distortions disappear and the equilibrium is characterized by the cost-plus-fee pricing feature (a similar result is obtained in Armstrong and Vickers 2001).[1] Both results are in stark contrast with the received wisdom in the nonlinear pricing literature (e.g., Mussa and Rosen 1978), where quality distortions occur for all types but $\overline{\theta}$ so as to reduce consumers' informational rents.

It is worth noting that in both papers, Rochet and Stole's analysis focuses on the case where the market is always fully covered on the vertical dimension, that is, the lowest (vertical) type of consumer ($\underline{\theta}$) is always served in the market. In particular, the main results in Rochet and Stole (2002) are derived under the condition $\underline{\theta}/\overline{\theta} \geq \frac{1}{2}$. In this paper, we focus on the case where the lowest vertical type of consumer is typically excluded from the market. More specifically, we assume that the vertical types of consumers are distributed uniformly over $[0, 1]$. This is a case not covered in Rochet and Stole because the condition $\underline{\theta}/\overline{\theta} \geq \frac{1}{2}$ is clearly violated. A direct consequence is that in our analysis, the minimal (vertical) type of consumer being served in the market is endogenously determined in equilibrium.

Interestingly, our findings are quite different from those in Rochet and Stole. In all the cases we analyze, the equilibrium exhibits perfect sorting (bunching never occurs),

[1] Rochet and Stole (2002) focus separately on the competitive regime and the monopoly regime (in terms of consumer coverage in the horizontal dimension). The mixed regime with both regimes present is analyzed in Rochet and Stole (1997).

and the quality distortion is maximal for the lowest type (we postpone a detailed discussion on the differences in our results from those of Rochet and Stole to Section 3). In fact, our results are more in line with those obtained in Mussa and Rosen (1978) and Maskin and Riley (1984). More importantly, focusing on the case where the lowest type of consumers being served is endogenously determined allows us to study the effect of varying horizontal differentiation (competition) on the market coverage, which is the main motivation of this paper. Our model is thus an extension of Rochet and Stole and our analysis complements that of Rochet and Stole.

The key to our analysis comes from the interaction between horizontal differentiation (competition) and screening on the vertical dimension. Although horizontal differentiation does not have a direct impact on the incentive compatibility (IC) conditions in the vertical dimension, it affects the IC conditions through the rent provisions to consumers.[2] This interaction in turn affects the menu of price–quality offers made by each brand (firm). It is through this interaction that we identify the effect of increased (horizontal) competition on the (vertical) market coverage of each firm.

Note that the interaction between the horizontal differentiation (competition) and screening in the vertical dimension is also present, though not explicitly mentioned, in Rochet and Stole. However, since the lowest (vertical) type covered is fixed, this interaction does not have an effect on market coverage in the vertical dimension in their model. On the other hand, since the lowest vertical type is endogenously determined in our model, there is one additional "freedom" for the working of this interaction: it now also affects market coverage in the vertical dimension.

Our base model includes both the monopoly and duopoly cases. In the duopoly case, there are two horizontally differentiated brands owned and operated by two separate firms. In the monopoly case, we assume that the elements of the model are the same as in the duopoly case except that the two brands are owned and operated by a single firm, the monopolist. This particular way of modeling provides a well-controlled benchmark; the difference in market structures is the only difference between the duopoly and monopoly cases. We focus on symmetric equilibria in which each firm (brand) makes the same menu of price–quality offers, and characterize the equilibrium menu of price–quality offers for both monopoly and duopoly. In both cases, the equilibrium menu of price–quality offers is unique, and a positive measure of consumers is excluded from the market. Moreover, the equilibrium price–quality offers in both cases exhibit perfect sorting.

Compared to the monopoly benchmark, we show that under duopoly more consumer types are covered and quality distortions decrease. This result is due to the interaction between horizontal competition and vertical screening. Intuitively, the competition in duopoly increases the rent provisions for higher type consumers, which relaxes the screening condition in the vertical dimension (informational rent consideration becomes less important, as higher type consumers obtain higher rent anyway due to competition). This leads to consumer types who were previously excluded being served in the market, and a reduction in quality distortions.

[2] As is standard in the screening literature, any IC contract can be represented by a rent provision schedule, which governs the utilities of consumers in equilibrium.

We study also how the degree of horizontal differentiation, or the intensity of competition, affects the equilibrium menu of offers. It turns out that the effects under the two market structures are quite different. Under monopoly, as the two brands become less differentiated, fewer consumer types are covered by each brand, and quality distortions become larger. The effects in the duopoly case are subtle. When the degree of horizontal differentiation (captured by the transportation cost k) is smaller than some cutoff value, a decrease in k results in more consumer types being served and smaller quality distortions; when k is larger than the cutoff value, a decrease in k results in fewer consumer types being served, and the effect on quality distortions is not uniform. Again these results are driven by the interplay between the horizontal differentiation and screening on the vertical dimension.

Finally, we extend our analysis of the duopoly model to a finite n-firm case and demonstrate that the analysis can be translated into that of the duopoly model by proper normalization. We show that an increase in the number of firms is equivalent to a decrease in k in the duopoly model. We thus conclude that when the initial competition level is not too low (n is large), an increase in the number of firms results in more consumer types being served by each firm and smaller quality distortions, while when the initial competition level is low (n is small), an increase in the number of firms results in fewer consumer types being served by each firm, though the effect on quality distortions is not uniform.

A number of other papers also study nonlinear pricing in competitive settings with both horizontally and vertically differentiated products—see, for example, Gilbert and Matutes (1993), Stole (1995), Verboven (1999), Villas-Boas and Schmidt-Mohr (1999), Ellison (2005), and Armstrong and Vickers (2001, 2006). However, these papers assume that all consumer types in the vertical dimension are served in the market. This full coverage assumption greatly simplifies the analysis, but precludes the effect of competition on consumer coverage in the vertical dimension, which is central to our analysis.

The paper is organized as follows. Section 2 introduces the base model with two brands. Section 3 derives the optimal symmetric menu of price–quality offers under monopoly. Section 4 characterizes the symmetric equilibrium in the duopoly model, and investigates how the equilibrium changes as the market structure moves from monopoly to duopoly. We extend our analysis to the arbitrary n-firm case in Section 5. Section 6 discusses the robustness of our analysis. Section 7 concludes.

2. The model

We consider a market with both vertically and horizontally differentiated products where consumers' preferences differ in two dimensions. In the horizontal dimension, consumers have different tastes over different brands (firms), while in the vertical dimension consumers have different marginal utilities over quality.[3] Although neither

[3]For example, the wholesale market for flat white cotton bed sheets of a particular size fits into this framework. Thread count would be the vertical attribute and the country of origin would be the horizontal attribute. Sellers are firms located in a single country, and buyers are brand-name distributors and department stores with house brands. See http://www.tradekey.com/ks-bed-sheet for more details of this market. We thank Edward Green for suggesting this example to us.

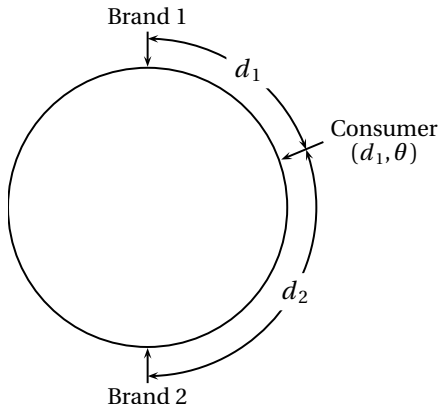

FIGURE 1. A two-brand base model.

type is observable to firms, in our model the single-crossing property is satisfied only in the vertical dimension. As a result firms can make offers to sort consumers only with respect to their vertical types.[4]

Our basic model studies the two-brand case under both duopoly and monopoly market structures. Under duopoly, two firms own two distinct brands, brand 1 and brand 2, respectively. Each firm (brand) offers a variety of vertically differentiated products, that is, goods of different qualities, which are indexed by q, $q \in \mathbb{R}_+$.[5] Quality q is both observable and contractible.

There is a continuum of consumers in the market, whose preferences differ on two dimensions: the "taste" dimension over the brands and the "quality" dimension. We model the taste dimension as the horizontal "location" of a consumer on a unit-length circle representing the ideal brand for that consumer.[6] As depicted in Figure 1, the locations of brands1 and 2 evenly split the circle. Let d_i be the distance between a consumer's location and brand i's location. Then d_i is this consumer's horizontal type, $i = 1, 2$. Because $d_1 + d_2 = \frac{1}{2}$, either d_1 or d_2 alone fully captures a consumer's preference over the two brands.

The consumers' varying preferences over the quality dimension are captured by θ, $\theta \in [0, 1]$, which we call a consumer's vertical type. A consumer is thus characterized by a two-dimensional type (d_i, θ) (either $i = 1$ or $i = 2$). Neither θ or d_i is observable by either firm. We assume that consumers are uniformly located along the unit-length circle, and the vertical types of consumers at each location are distributed uniformly over the unit interval: $\theta \sim U[0, 1]$. A consumer's horizontal location and vertical type are independent.

[4]For this reason our paper does not belong to the multi-dimensional screening literature (e.g., Laffont et al. 1987, McAfee and McMillan 1988, Armstrong 1996, and Rochet and Choné 1998).

[5]Throughout "quality" should be interpreted as a summary measure for a variety of product characterisitics, such as safety, reliability, and durability.

[6]For two brands, it is sufficient to use a unit interval. We work with a unit-length circle since doing so makes it easier to extend our model to the arbitrary n-brand or n-firm case later.

Each consumer demands at most one unit of a good. If a type-(d_i, θ) consumer purchases one unit of the brand-i product with quality q at price t, her utility is given by

$$u(q, t, d_i, \theta) = \theta q - t - k d_i \tag{1}$$

where k, $k > 0$, can be interpreted as the per unit "transportation" cost. Note that the smaller is k, the less horizontally differentiated are the two brands. The reservation utility of a consumer who purchases no product is normalized to be 0.

We assume that the two brands (firms) have the same production technology. Specifically, to produce a unit of quality-q product a firm incurs a cost $c(q) = q^2/2$. Thus, each firm (brand) has a per-customer profit function given by

$$\pi(t, q) = t - q^2/2. \tag{2}$$

Each firm makes a menu of price–quality offers, which is a collection of price and quality pairs. Given the menus of price–quality offers made by both firms (brands), consumers decide whether to make a purchase, and if so, which brand to choose and which offer to accept. It is well known that in the environment of competitive nonlinear pricing, it is no longer without loss of generality to restrict attention to direct contracts.[7] To sidestep this problem, as in Rochet and Stole (2002) we restrict attention to deterministic contracts.[8] Since the preferences of a consumer with vertical type θ over the available price–quality pairs conditional on purchasing from a firm (brand) are independent of her horizontal type d_i, in what follows it is without loss of generality to consider direct contracts (offers) of the form $\{q(\theta), t(\theta)\}_{\theta \in [0,1]}$. For brevity of exposition, we often refer to vertical types simple as types, especially when there is no confusion in the context.

Our solution concept is Bertrand–Nash equilibrium: given the other firm's menu of offers, each firm's menu of offers maximizes its expected total profit.

This basically completes a description of the duopoly model. For the monopoly model, our main goal is to lay down a benchmark with which we can identify the effect of competition on the menu of offers. As such in the monopoly model we need to control for all but the market structure. We thus assume that in the monopoly case, all the elements of the model are the same as in the duopoly model, except that the two brands are now owned and operated by the same firm, which is the monopolist.[9] The

[7]As demonstrated in a series of examples in Martimort and Stole (1997) and Peck (1997), equilibrium outcomes in indirect mechanisms may not be supported when sellers are restricted to using direct mechanisms where buyers report only their private types. Moreover, as demonstrated by Martimort and Stole (1997), an equilibrium in such direct mechanisms may not be robust to the possibility that sellers might deviate to more complicated mechanisms. The reason for such failures, as pointed out by McAfee (1993) and Katz (1991), is that in competition with nonlinear pricing the offers made by other firms may also be the private information of the consumers when they make their purchase decisions, which means that this private information can also potentially be used when firms set up their revelation mechanisms.

[8]See Rochet and Stole (2002) for a discussion of the restrictions resulting from focusing on deterministic contracts. More general approaches to restoring the "without loss of generality" implication of the revelation principle in the environment of competitive nonlinear pricing are proposed and developed by, for example, Epstein and Peters (1999), Peters (2001), and Page and Monteiro (2003).

[9]So our benchmark is a multi-product monopoly, which has an alternative interpretation as being a collusive duopoly.

monopolist's objective is to maximize the joint profits from the two brands by choosing the menu of offers for each brand.

As an analytical benchmark, given (1) and (2), the first-best (efficient) quality provision is $q^*(\theta) = \theta$. We can thus define $\theta - q(\theta)$ as the *quality distortion* for type θ given the quality schedule q.

Incentive compatible price-quality offers

Let $U_i(\hat{\theta}, \theta, d_i)$ be the utility obtained by a consumer of type (θ, d_i) who reports $\hat{\theta}$ and purchases a unit of brand i's product. Then

$$U_i(\hat{\theta}, \theta, d_i) = \theta q_i(\hat{\theta}) - t_i(\hat{\theta}) - k d_i. \tag{3}$$

Incentive compatibility requires

$$\forall(\theta, \hat{\theta}) \in [0,1]^2,\ U_i(\theta, \theta, d_i) \geq U_i(\hat{\theta}, \theta, d_i) \text{ for } i = 1,2. \tag{4}$$

Since (3) satisfies the single-crossing property in (θ, q_i), we can show the following "constraint simplification" lemma.

Lemma 1. *The IC condition (4) is satisfied if and only if the following two conditions hold.*

(i) $U_i(\theta, \theta, d_i) = \int_{\theta_i^*}^{\theta} q_i(\tau)\, d\tau - k d_i$ *for all* $\theta \geq \theta_i^*$ *and* $i = 1,2$

(ii) $q_i(\theta)$ *is increasing in* θ

where $\theta_i^* \in [0,1)$ *is the lowest type that purchases from brand* i.

Lemma 1 is a standard result in the one-dimensional screening literature. This also applies to our model because the consumers' utility functions are separable in q and d_i. Here θ_i^* can be regarded as a separate choice variable for brand i: any consumer whose type is below θ_i^* is excluded from the market for brand i. Alternatively, one can think of brand i making a null offer ($q_i = 0$ and $t_i = 0$) to all consumers whose types are below θ_i^*. Define

$$y_i(\theta) = \int_{\theta_i^*}^{\theta} q_i(\tau)\, d\tau,\ i = 1,2. \tag{5}$$

Then by Lemma 1, $y_i(\theta)$ is the *rent provision* to the type-$(\theta, 0)$ consumer specified by the menu of IC offers made by brand i. The equilibrium utility enjoyed by a consumer of type (θ, d_i) can now be written as $y_i(\theta) - k d_i$. Moreover, the quality and price specified in the original offer can be recovered from $y_i(\theta)$ as follows:

$$q_i(\theta) = y_i'(\theta) \text{ and } t_i(\theta) = \theta q_i(\theta) - y_i(\theta).$$

Thus any menu of IC offers can be characterized by the rent provision schedules (y_i, $i = 1,2$).[10]

[10] In this regard we follow the lead of Armstrong and Vickers (2001), who model firms as supplying utility directly to consumers.

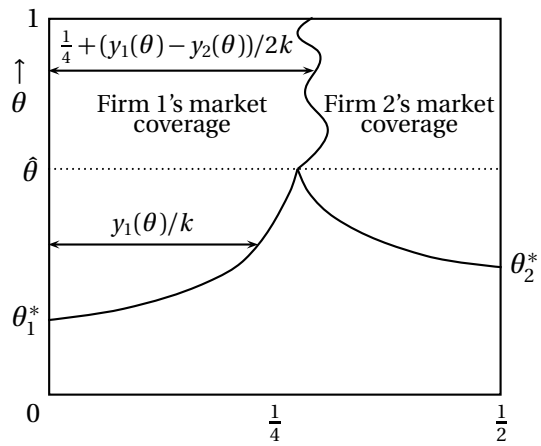

FIGURE 2. An illustration of market shares and market coverage.

Individual rationality and market shares

Given rent provision schedules $\{y_i(\theta)\}$, $i = 1,2$, each consumer decides whether to make a purchase, and if he does, what product (brand and quality) to purchase. If a consumer of type (θ, d_i) chooses to purchase a product from brand i, then we must have

$$y_i(\theta) - kd_i \geq \max\{0, y_{-i}(\theta) - k(\tfrac{1}{2} - d_i)\}.$$

Alternatively, we have

$$d_i \leq \min\left\{\frac{y_i(\theta)}{k}, \frac{1}{4} + \frac{1}{2k}(y_i(\theta) - y_{-i}(\theta))\right\} := s_i(\theta). \tag{6}$$

The number $2s_i(\theta)$ is the total measure of type-θ consumers who purchase brand i products. Figure 2 illustrates one half of the market share for each brand (the other half not shown is symmetric).

From Figure 2, we can see that there is a cutoff type $\hat{\theta}$ above which the market is fully covered (consumers are served regardless of their horizontal locations), and below which the market is not fully covered. This is because $y_i(\theta)$ is increasing in θ by (5). Under duopoly, the full coverage range $[\hat{\theta}, 1]$ can also be called *the competition range* since the two firms are competing for customers over this range, and the partial coverage range $[\theta_i^*, \hat{\theta})$ can also be called *the local monopoly range*. Note that $\hat{\theta}$ is endogenously determined by the condition

$$y_1(\hat{\theta}) + y_2(\hat{\theta}) = \tfrac{1}{2}k.$$

Given y_{-i}, brand i's total expected profit is twice

$$\int_{\theta_i^*}^{1} [t_i(\theta) - \tfrac{1}{2}q_i^2(\theta)] s_i(\theta)\, d\theta = \int_{\theta_i^*}^{1} \left[\theta q_i(\theta) - y_i(\theta) - \tfrac{1}{2}q_i^2(\theta)\right] s_i(\theta)\, d\theta. \tag{7}$$

By separating the partial coverage range from the full coverage range, we can rewrite (7) as the sum of two integrations:

$$\begin{aligned}\int_{\theta_i^*}^{\hat{\theta}} \left[\theta q_i(\theta) - y_i(\theta) - \tfrac{1}{2}q_i^2(\theta)\right] \frac{y_i(\theta)}{k}\, d\theta \\ + \int_{\hat{\theta}}^{1} \left[\theta q_i(\theta) - y_i(\theta) - \tfrac{1}{2}q_i^2(\theta)\right] \cdot \left[\frac{1}{4} + \frac{1}{2k}(y_i(\theta) - y_{-i}(\theta))\right] d\theta.\end{aligned} \tag{8}$$

The maximization of (8) subject to the transition equation $y_i'(\theta) = q_i(\theta)$ and the corresponding endpoint conditions can be viewed as an optimal control problem with two potential phases.[11] What makes it different from an ordinary single-phase optimal control problem is that now we need to solve also for the optimal switching "time" $\hat{\theta}$, at which the first phase switches to the second phase.

3. Monopoly

Under monopoly, the two brands are owned by a single firm. The monopolist's objective is to maximize the joint profit from the two brands. Since consumers are uniformly distributed along the horizontal dimension and the two brands' production technologies are symmetric, we focus on the symmetric solution in which each brand makes the same menu of offers and the resulting market shares are symmetric.[12] We can thus drop the subscripts to write $y_i(\theta) = y(\theta)$, $i = 1, 2$. Simplifying (6), the market share becomes $s_i(\theta) = s(\theta) = \min\left\{y(\theta)/k, \frac{1}{4}\right\}$.

The monopolist's problem can be formulated as

$$\begin{aligned}\max \int_{\theta^*}^{\hat{\theta}} \left[\theta q(\theta) - y(\theta) - \tfrac{1}{2}q^2(\theta)\right] \frac{y(\theta)}{k}\, d\theta + \int_{\hat{\theta}}^{1} \left[\theta q(\theta) - y(\theta) - \tfrac{1}{2}q^2(\theta)\right] \tfrac{1}{4}\, d\theta \\ \text{s.t. } y'(\theta) = q(\theta), q'(\theta) \geq 0 \\ y(\hat{\theta}) = k/4, y(\theta^*) = 0,\end{aligned}$$

where θ^* is the lowest type of consumer served, that is, $y(\theta^*) = 0$,[13] and $\hat{\theta}$ is the unique solution to $y(\theta)/k = \frac{1}{4}$.

As is standard in the literature, we solve the relaxed program by dropping the constraint $q'(\theta) \geq 0$ (the monotonicity of $q(\theta)$ is verified later to justify this approach).

[11] An early application of the two-phase optimal control technique can be found in Amit (1986), who considers a petroleum recovery process that has two potential phases with different technologies yielding different extraction rates.

[12] We focus on the symmetric solution here for ease of comparison with the duopoly case, where we focus on symmetric equilibrium in which each firm makes the same menu of offers. While a formal proof is not attempted here, we conjecture that any optimal solution for the monopolist is symmetric.

[13] If $y(\theta^*) > 0$, then for some sufficiently small ϵ, it can be verified that some type-$(\theta^* - \epsilon)$ consumers prefer accepting the offer $y(\theta^*)$ to staying out of the market, which contradicts the assumption that θ^* is the lowest type being served.

Define the Hamiltonian function of the two phases as follows:

$$H = \begin{cases} H_1 = \left[\theta q - y - \frac{1}{2}q^2\right]\dfrac{y}{k} + \lambda q & \text{if } \theta^* \leq \theta < \hat{\theta} \\ H_2 = \left[\theta q - y - \frac{1}{2}q^2\right]\frac{1}{4} + \lambda q & \text{if } \hat{\theta} < \theta \leq 1. \end{cases}$$

It can be verified that Phase I (the partial coverage range) is characterized by the following differential equation:

$$3y - \tfrac{1}{2}y'^2 - yy'' = 0. \tag{9}$$

Given the lower endpoint condition $y(\theta^*) = 0$, it can be verified that the unique solution to (9) is given by[14]

$$y(\theta) = \tfrac{3}{4}(\theta - \theta^*)^2,\ q(\theta) = \tfrac{3}{2}(\theta - \theta^*).$$

Similarly, in phase II (the full coverage range) we can obtain the differential equation $y'' = 2$. Combined with the transversality condition $\lambda(1) = 0$, the solution to this system is given by

$$y(\theta) = \theta^2 - \theta + \beta,\ q(\theta) = 2\theta - 1,$$

where β is a parameter yet to be determined. Note that in both phases $q(\theta)$ is increasing in θ. Thus the solutions to the relaxed program are also the solutions to the original program. Moreover, since in both phases $q(\theta)$ is strictly increasing in θ, the optimal menu of offers exhibits perfect sorting.

To determine $\hat{\theta}$, we apply smooth pasting: $y(\hat{\theta}^-) = y(\hat{\theta}^+)$ and $q(\hat{\theta}^-) = q(\hat{\theta}^+)$.[15] We thus have

$$\begin{aligned} \tfrac{3}{4}(\hat{\theta} - \theta^*)^2 &= \hat{\theta}^2 - \hat{\theta} + \beta = \tfrac{1}{4}k \\ \tfrac{3}{2}(\hat{\theta} - \theta^*) &= 2\hat{\theta} - 1. \end{aligned}$$

Given all these equations, we can solve for $\hat{\theta}$, θ^*, and β as follows:

$$\theta^{*M} = \tfrac{1}{2} - \tfrac{1}{12}\sqrt{3k} \tag{10}$$

$$\hat{\theta}^M = \tfrac{1}{2} + \tfrac{1}{4}\sqrt{3k} \tag{11}$$

$$\beta = \tfrac{1}{4} + \tfrac{1}{16}k.$$

It is easily verified that $\hat{\theta}$ has an interior solution only when $k < \frac{4}{3}$. If $k \geqslant \frac{4}{3}$, we have the corner solution $\hat{\theta} = 1$. That is, if $k \geqslant \frac{4}{3}$ phase II is never entered (there is no interaction between the two brands). In that case we can use the transversality condition, $\lambda(1) = 0$, to pin down $\theta^{*M} = \frac{1}{3}$. The above analysis is summarized in the next result.

[14]The uniqueness is implied in Rochet and Stole (2002, appendix, p. 304): if a convex solution to the differential equation (9) exists for a given set of boundary conditions, it is unique.

[15]Smooth pasting is a consequence of the Weierstrass–Erdmann necessary condition.

Proposition 1. *In the monopoly model, the optimal symmetric menu of offers is unique and exhibits perfect sorting. Specifically, for* $k \in \left(0, \frac{4}{3}\right)$,

$$y(\theta) = \begin{cases} \frac{3}{4}(\theta - \theta^{*M})^2 & \text{if } \theta^{*M} \leq \theta \leq \hat{\theta}^M \\ \theta^2 - \theta + \frac{1}{4} + \frac{1}{16}k & \text{if } \hat{\theta}^M < \theta \leq 1, \end{cases}$$

where θ^{*M} *and* $\hat{\theta}^M$ *are given by (10) and (11), respectively. For* $k \geqslant \frac{4}{3}$,

$$y(\theta) = \frac{3}{4}\left(\theta - \frac{1}{3}\right)^2, \theta \in \left[\frac{1}{3}, 1\right].$$

The optimal menu of offers exhibits several salient features. First, there is always a positive measure of types of consumers (regardless of the horizontal location) who are excluded from the market ($\theta^{*M} > 0$). The underlying reason for the exclusion involves a consideration of the informational rent. Making offers to all types may increase the firm's profit from types in $[0, \theta^{*M})$. However, doing so necessarily increases the informational rent to all types above θ^{*M} due to the screening condition (5), which reduces the firm's profit from those types. The optimal θ^{*M}, which balances these two opposing effects, should thus be strictly positive. Second, there is quality distortion for all but the highest type consumers, i.e., $q(\theta) < \theta$ for all $\theta \in [\theta^{*M}, 1)$. This is again driven by the informational rent. Finally, the optimal offers exhibit perfect sorting. That is, different types of consumers choose different offers. Our results thus imply that bunching does not occur and the quality provision for the lowest type covered is always distorted downwards. These are very different from the results obtained by Rochet and Stole (1997, 2002), who show that either bunching occurs at a lower interval, or perfect sorting occurs with efficient quality provision for the lowest type.

This difference between our results and theirs at first appears puzzling, given that the differential equation (9) is either the same as the one characterized in the monopolistic regime of Rochet and Stole (1997) or a special case of the Euler equation in the monopoly case of Rochet and Stole (2002). The key to solving the puzzle is to observe the difference in boundary conditions. Note that the ratio of the lowest type to the highest type $\gamma = \underline{\theta}/\overline{\theta}$ is assumed to be greater than 0.76 in Rochet and Stole (1997) and greater than 0.5 in Rochet and Stole (2002). Both conditions imply that all the (vertical) types are covered. As a result, the state variable y is free at the lowest type $\underline{\theta}$, which gives rise to the boundary condition $\lambda(\underline{\theta}) = 0$. Substituting this into the first order condition $\partial H_1/\partial q = 0$ yields $q(\underline{\theta}) = \underline{\theta}$. In other words, we have efficient quality provision at the bottom if the monotonicity constraint on q is satisfied (the perfect sorting case).[16] Note also that sorting can become quite costly for the monopolist given the requirement of no quality distortion at $\underline{\theta}$, which explains why bunching may occur at a lower interval starting from $\underline{\theta}$. On the other hand, in our model the lowest possible type $\underline{\theta}$ is 0 ($\gamma = 0$), thus not all types are covered and the lowest type covered, θ^*, is endogenously determined. This leads to a different set of boundary conditions: $y(\theta^*) = 0$ and $H(\theta^*) = 0$. Combined

[16]This result of efficiency at the bottom does not hold in the discrete setting. In their appendix, Rochet and Stole (2002) demonstrate that the distortion at the bottom decreases as the type space becomes finer, and completely disappears in the limit as θ is distributed continuously.

with the differential equation (9), these conditions pin down a unique perfect sorting solution in which $q(\theta^*) = 0$.[17]

Thus in a sense our analysis is complementary to that in Rochet and Stole: while they study the case with full coverage of vertical types (γ is big), we analyze the case with endogenously determined coverage of vertical types (γ is small). It is worth noting that the two cases lead to qualitatively different results. To better understand the link between our results and those of Rochet and Stole, fix the upper bound of the vertical type, $\overline{\theta}$, and assume that θ is now distributed uniformly over $[\underline{\theta}, \overline{\theta}]$. First we ignore the constraint $\theta^* \geq \underline{\theta}$. Following exactly the same derivations that lead to Proposition 1, we can verify that $\theta^* = \frac{1}{2}\overline{\theta} - \frac{1}{12}\sqrt{3k} \equiv \theta_c$. (If $\underline{\theta} \leq \theta_c$ then the constraint $\theta^* \geq \underline{\theta}$ is not binding and our approach is justified.) When $\underline{\theta}$ is 0, our results apply: there is an endogenously determined lowest type covered, θ^*, with perfect sorting and $q(\theta^*) = 0$. This feature stays the same until $\underline{\theta}$ is raised just above θ_c. When $\underline{\theta}$ is just above θ_c, the constraint $\theta^* \geq \underline{\theta}$ is binding and the case of Rochet and Stole applies since all the vertical types are covered.[18] When $\underline{\theta} = \theta_c$, if the monotonicity constraint does not bind, then the boundary condition requires efficient quality provision at $\underline{\theta}$.[19] But continuity implies that the optimal solution should not change drastically at $\underline{\theta} = \theta_c$. Thus monotonicity must fail, leading to bunching at the lower end near $\underline{\theta}$. Intuitively, when $\underline{\theta}$ is slightly above θ_c (γ is relatively small), efficient quality provision at $\underline{\theta}$ is costly since it increases the informational rent for all higher types, the measure of which is big since γ is relatively small. Optimality thus requires bunching. As $\underline{\theta}$ is further raised close to $\overline{\theta}$ (γ becomes big enough), efficient quality provision at $\underline{\theta}$ becomes less costly since there are fewer higher types. As a result, the monotonicity constraint is more likely to be satisfied even with efficient quality provision at $\underline{\theta}$. Therefore, perfect sorting is more likely when γ is big. This, we believe, explains why in Rochet and Stole the solution involves perfect sorting when γ is sufficiently large.

We are interested in how the degree of horizontal differentiation, which is parameterized by k, affects the market coverage and quality distortions. Equation (10) shows that for $k \in (0, \frac{4}{3})$, θ^{*M} is decreasing in k, and for $k \geqslant \frac{4}{3}$, $\theta^{*M} = \frac{1}{3}$ is independent of k. Thus when two brands become more horizontally differentiated (bigger k), more consumer types are served by the monopolist. From the equilibrium quality schedules it can also be seen that quality distortions become smaller in Phase I but are unaffected in Phase II. We summarize these results in the following proposition.

Proposition 2. *In the monopoly model, when two brands become more horizontally differentiated, more consumer types are served and quality distortions become smaller in the partial coverage range and remain unaffected in the full coverage range.*

[17] It can be easily verified that the quadratic functional form solution, which works in our case, does not satisfy the differential equation system in Rochet and Stole, simply because it violates their boundary conditions.

[18] If $\gamma = \underline{\theta}/\overline{\theta} \geq \frac{1}{2}$ as assumed in Rochet and Stole (2002), then $\underline{\theta} \leq \theta_c$ is violated. In that case our approach of ignoring the constraint $\theta^* \geq \underline{\theta}$ is not justified, and the solution may not be perfect sorting, which is consistent with Rochet and Stole's finding.

[19] This implies that $\lim_{\underline{\theta} \to \theta_c^+} q(\theta^*) = \theta_c$ while $\lim_{\underline{\theta} \to \theta_c^-} q(\theta^*) = 0$.

To understand the intuition for this result, we first need to understand the effects on profit of increasing the rent provision. Raising rent provisions (hence the total rent) to consumers has two effects. The first is to reduce the firm's profitability per consumer (which can be termed the *marginal effect*), and the second is to attract more consumers (which can be termed the *market share effect*). Thus profit maximization requires an optimal balance between these two opposing effects. Note that with asymmetric information, the firm cannot freely vary the rent provision for certain types of consumers without affecting the rent provisions to other types. That is, rent provisions can only be adjusted subject to the screening condition, (5), which implies that changing the rent provision for some type will affect the rent provisions for all the types above. Hence the optimal rent provision schedule reflects an optimal trade-off between the marginal effect and market share effect subject to the screening condition.

In view of this insight, it is now straightforward to think through the intuition behind Proposition 2. As k increases, by fixing the previous menu of offers (holding y fixed), $\hat{\theta}$ increases and $y(\theta)/k$ decreases, which implies that the market shares in both the full and partial coverage ranges shrink. To counter this effect, the monopolist has an incentive to increase $y(\theta)$ in an attempt to partially restore the loss of the market shares. By the screening condition (5), this can be achieved by either moving the schedule q upward or pushing θ^{*M} downward, and both occur in equilibrium. Hence Proposition 2 is driven by an interaction between horizontal differentiation and screening in the vertical dimension, which occurs through the rent provision schedule $y(\theta)$.

4. Duopoly

In the duopoly model, each firm's objective is to maximize its profit by choosing a menu of offers, given the other firm's menu of offers. Since both firms are symmetric in terms of their production technology and market positions, we focus on *symmetric equilibrium*, in which each firm makes the same menu of offers, hence the same rent provision schedule $y^*(\theta)$, $\theta \in [\theta^{*D}, 1]$ (θ^{*D} is the lowest type that is served in the market). Formally, the pair (y^*, y^*) constitutes a Bertrand–Nash equilibrium if given $y_{-i}(\theta) = y^*(\theta)$, $\theta \in [\theta^*, 1]$, firm i's best response is to choose $y_i(\theta) = y^*(\theta)$, $\theta \in [\theta^*, 1]$, as well.

Given the two firms' rent provision schedules $y_1(\theta)$ and $y_2(\theta)$, the consumers' type space is demarcated into two ranges: the competition range ($\theta > \hat{\theta}$) and the local monopoly range ($\theta < \hat{\theta}$). The switching point $\hat{\theta}$ is determined by $y_i(\hat{\theta}) = k/2 - y_{-i}(\hat{\theta})$.

Suppose $y_{-i}(\theta) = y^*(\theta)$, $\theta \in [\theta^*, 1]$. Then firm i's relaxed program (by ignoring the monotonicity of q_i) is

$$\begin{aligned}
\max \int_{\theta_i^*}^{\hat{\theta}} & [\theta q_i(\theta) - y_i(\theta) - c(q_i(\theta))] \frac{y_i(\theta)}{k}\, d\theta \\
& + \int_{\hat{\theta}}^{1} [\theta q_i(\theta) - y_i(\theta) - c(q_i(\theta))] \cdot \left[\frac{1}{4} + \frac{1}{2k}(y_i(\theta) - y^*(\theta))\right] d\theta \\
\text{subject to} \quad & y_i'(\theta) = q_i(\theta), y_i(\theta_i^*) = 0,\ \theta_i^* \text{ free} \\
& y_i(\hat{\theta}) = \frac{k}{2} - y^*(\hat{\theta}),\ \hat{\theta} \text{ free},\ y_i(1) \text{ free}.
\end{aligned}$$

We define the Hamiltonian function as follows:

$$H=\begin{cases} H_1=\left[\theta q_i-y_i-\frac{1}{2}q_i^2\right]y_i/k+\lambda q_i & \text{if } \theta_i^*\le\theta<\hat{\theta} \\ H_2=\left[\theta q_i-y_i-\frac{1}{2}q_i^2\right]\cdot[\frac{1}{4}+(1/2k)(y_i(\theta)-y^*(\theta))]+\lambda q_i & \text{if } \hat{\theta}<\theta\le 1. \end{cases}$$

For phase I ($\theta<\hat{\theta}$), we can follow exactly the same steps as in the monopoly model to obtain

$$y^*(\theta)=\tfrac{3}{4}(\theta-\theta^*)^2,\ q^*(\theta)=\tfrac{3}{2}(\theta-\theta^*).$$

For phase II ($\theta>\hat{\theta}$), the optimality condition and the costate equation evaluated at $y_i=y^*$ are given by

$$\begin{aligned} 0&=(\theta-q^*)\tfrac{1}{4}+\lambda \\ \lambda'&=\frac{1}{4}-\frac{1}{2k}\left[\theta q^*-y^*-\tfrac{1}{2}q^{*2}\right]. \end{aligned}$$

After eliminating λ from the above equations we obtain the differential equation

$$y^{*\prime\prime}=2-\frac{2}{k}\left(\theta y^{*\prime}-y^*-\tfrac{1}{2}y^{*\prime 2}\right). \tag{12}$$

Letting $y_i=y_{-i}=y^*$, the switching point $\hat{\theta}$ is defined by $y^*(\hat{\theta})=\frac{1}{4}k$. Applying smooth pasting for both y^* and q^* at $\hat{\theta}$, we have $\hat{\theta}-\theta^*=\sqrt{k/3}$. From the Phase I solution, we can obtain $y^{*\prime}(\hat{\theta})=q^*(\hat{\theta})=\sqrt{3k}/2$. Finally $\lambda(1)=0$ implies that $y^{*\prime}(1)=q^*(1)=1$.

Now the existence of a symmetric equilibrium boils down to the existence of $\hat{\theta}\in(0,1]$ and a convex function y^* defined over $[\hat{\theta},1]$ that satisfy the following equations (we drop the superscripts to simplify notation):[20]

$$\begin{aligned} y''&=2-(2/k)\left(\theta y'-y-\tfrac{1}{2}y'^2\right) \\ y(\hat{\theta})&=k/4 \\ y'(\hat{\theta})&=\sqrt{3k}/2 \\ y'(1)&=1. \end{aligned} \tag{13}$$

Proposition 3. *For $k\in\left(0,\frac{4}{3}\right)$, the duopoly model has a unique symmetric equilibrium, which exhibits perfect sorting and is given by*

$$y(\theta)=\begin{cases} \frac{3}{4}(\theta-\theta^{*D})^2 & \text{if } \theta^{*D}\le\theta\le\hat{\theta}^D \\ y^*(\theta) & \text{if } \hat{\theta}^D\le\theta\le 1, \end{cases}$$

where $(\hat{\theta}^D,y^*(\theta))$ is the unique solution to the system (13) and $\theta^{*D}=\hat{\theta}^D-\sqrt{k/3}$.

For $k\ge\frac{4}{3}$, the duopoly equilibrium is the same as the monopoly outcome:

$$y(\theta)=\tfrac{3}{4}\left(\theta-\tfrac{1}{3}\right)^2,\ \theta\in\left[\tfrac{1}{3},1\right].$$

[20]We need $y''(\theta)\geqslant 0$ to ensure $q'(\theta)\ge 0$.

This result is proved in the Appendix. In the proof we show that given $k \in \left(0, \frac{4}{3}\right)$, the solution to the differential equation system (13) exists and is unique. Moreover, $y^*(\theta)$ is strictly convex. The system (13) is not a standard ordinary differential equation (ODE) system partly due to the fact that the boundary conditions involve an endogenously determined endpoint ($\hat{\theta}$). Thus no existing ODE theorem can be directly applied to show the existence and uniqueness of a solution. The proof is somewhat tedious and hence relegated to the Appendix. It is clear that the system (13) has no closed-form solution. So the schedule $y^*(\theta)$ can be obtained only from numerical computations.

Armstrong and Vickers (2001) and Rochet and Stole (2002) demonstrate that in a market where consumers are fully covered on both horizontal and vertical dimensions, there are no quality/quantity distortions by competing duopolists. The intuition seems to be that the competitive pressure induces a type of Ramsey pricing by the firms, i.e., any inefficient offer could be dominated by making a more efficient offer along with a more profitable fixed fee. Proposition 3, however, suggests that this conclusion is no longer valid in a setting with partial market coverage. When the marginal utilities over quality are sufficiently low for some consumers, each competing duopolist becomes a local monopolist for those types. It thus becomes profitable to exclude some of these types from the market. This endogenously determined threshold then induces distortions for many infra-marginal consumers.[21]

Let q_D and q_M be the equilibrium quality provision schedules in the duopoly model and monopoly model, respectively. Despite the absence of a closed-form solution in the duopoly model, we are able to rank θ^{*D} and θ^{*M} and the schedules q_D and q_M unambiguously.

Proposition 4. *Given $k \in \left(0, \frac{4}{3}\right)$ we have $\theta^{*D} < \theta^{*M}$ and $q_D(\theta) > q_M(\theta)$ for $\theta \in [\theta^{*D}, 1)$, which implies that compared to the monopoly benchmark, more consumer types are served by each firm, and quality distortions are smaller in duopoly equilibrium.*

This result is proved in the Appendix. The result is shown by comparing the differential equation systems under the two market structures. Figure 3 compares the market coverages under duopoly and monopoly. Since $\theta^{*D} < \theta^{*M}$ and $q_D(\theta) > q_M(\theta)$, it is easily seen that $y_D(\theta) > y_M(\theta)$, which in turn implies that the market coverage area under duopoly contains that under monopoly.

To see the intuition behind this comparison result, start by assuming that in the duopoly case each firm makes the same optimal symmetric menu of offers as in the monopoly case. As a result the partial coverage and full coverage ranges are the same under both market structures. Note that in the full coverage range ($\theta \in [\hat{\theta}^M, 1]$), the market share effect is absent under monopoly since the market is fully covered and the "competition" between the two brands is internalized by the monopolist; however, under duopoly the market share effect is present since each firm (brand) tries to steal the other firm's market share. Thus the market share effect is stronger under duopoly, and

[21] Rochet and Stole (1997) have a similar finding in their analysis of the mixed regime, where consumers are not fully covered along the horizontal dimension. However, the efficiency at the bottom ($\underline{\theta}$) still persists in their analysis, which highlights another difference between our approach and theirs.

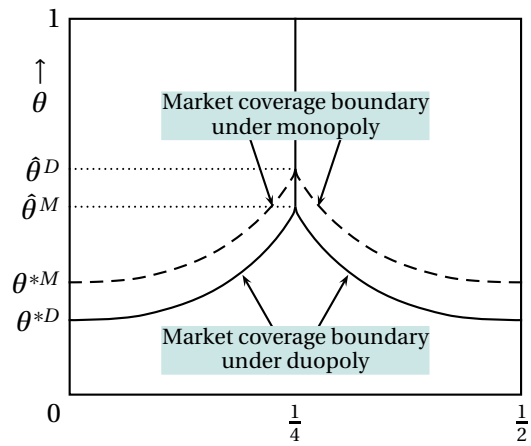

FIGURE 3. Duopoly vs. Monopoly

each firm (brand) has an incentive to increase rent provision. Therefore moving from monopoly to duopoly, $\theta^{*D} < \theta^{*M}$ and $q_D(\theta) > q_M(\theta)$ (by the screening condition (5)). Another way to see this is that competition under duopoly increases rent provisions to higher-type consumers (served in the full coverage range), which relaxes the screening condition in the vertical dimension: under duopoly firms worry less about providing additional (informational) rent for the higher-type consumers, as the higher-type consumers are going to enjoy higher rent anyway due to competition. Consequently those consumers not served under monopoly may be served under duopoly, and quality distortions become smaller.

Proposition 4 establishes that quality provision ($q(\theta)$) and market coverage are both larger under duopoly. It is thus not clear whether the average quality of products is also greater under duopoly. The answer is affirmative as indicated by the following proposition.

PROPOSITION 5. *The average quality of products offered under duopoly is higher than that under monopoly if* $k \in \left(0, \frac{4}{3}\right)$.

This result is proved in the Appendix. Intuitively speaking, competition leads to higher average quality for the following reasons. First, in the partial coverage range the average quality and the total measure of consumers covered are the same under monopoly and duopoly. Second, in the full coverage range the average quality is higher under duopoly since competition leads to smaller quality distortion. Finally, under duopoly the full coverage range covers more consumers than it does under monopoly. Since the average quality in the full coverage range is higher than that in the partial coverage range, this also contributes to a higher (overall) average quality under duopoly.[22]

[22]It would be desirable to study the effect of competition on the prices. However, no general conclusion can be drawn on this. For a offer that is targeted to a particular type, a direct effect of introducing com-

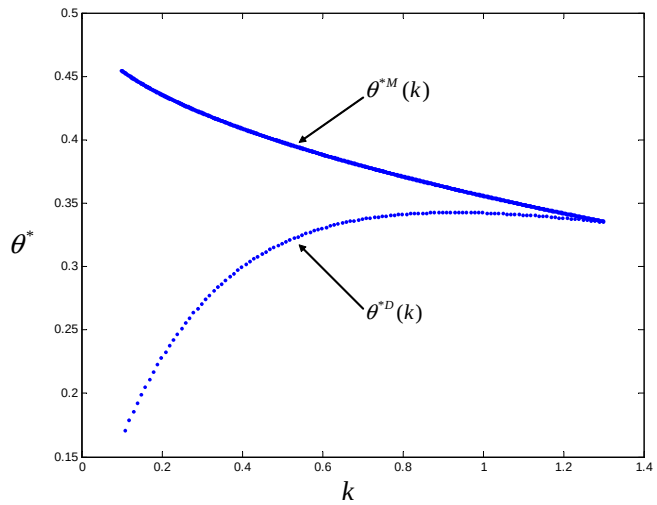

FIGURE 4. Comparison of participation thresholds.

As in the monopoly case, we are interested also in how changes in k affect the market coverage by each firm and the quality distortions. For convenience of comparison, we show the schedules of both θ^{*D} and θ^{*M} against k in Figure 4, where the schedule of θ^{*D} is plotted from numerical computation.

As can be seen from the figure, θ^{*M} is always decreasing as k increases. But for the duopoly model, there is a cutoff k^* such that for $k \in (0, k^*)$, θ^{*D} is increasing in k, and for $k \in \left(k^*, \frac{4}{3}\right)$, θ^{*D} is decreasing in k (for $k \geq \frac{4}{3}$, $\theta^{*D} = \theta^{*M} = \frac{1}{3}$ is independent of k). Our computation shows that the turning point k^* is approximately 0.91. Note that the decreasing trend of θ^{*D} in the range of $\left(k^*, \frac{4}{3}\right)$ is not quantitatively significant; in this range of k, θ^{*D} is in the range $[0.33, 0.35]$. On the other hand, the increasing trend of θ^{*D} in the range of $(0, k^*)$ is quantitatively significant; when $k = k^*$, θ^{*D} equals to 0.35, while as k converges to 0, θ^{*D} converges to 0 as well. The following comparative statics result is obtained from numerical computations.[23]

PROPOSITION 6. *In the duopoly case, when $k \in (0, k^*)$, as k decreases more consumer types are covered by each firm and quality distortions become smaller; when $k \in \left(k^*, \frac{4}{3}\right)$, as k decreases fewer consumer types are covered by each firm, and the effect on quality distortions is not uniform: there is a cutoff type, say $\widetilde{\theta}$, such that when $\theta \in [0, \widetilde{\theta})$, quality distortions become bigger, while when $\theta \in (\widetilde{\theta}, 1)$, quality distortions become smaller; when $k \geqslant \frac{4}{3}$, both firms are local monopolists, hence k affects neither the market coverage nor the quality distortions.*

petition is to decrease the price. However, an indirect effect is that this type gets a higher quality under competition, which tends to increase the price. The net effect is ambiguous.

[23]The MATLAB code for all the computations in this paper is available in a supplementary file on the journal website, http://econtheory.org/supp/336/supplement.txt.

Thus the effects of changing k on θ^* and quality distortions in the duopoly case are dramatically different from those in the monopoly benchmark. The intuitions spelled out previously continue to help, though the details are a bit more subtle. Under duopoly, a lower k implies not only less horizontal differentiation, but also more fierce competition between the two firms.

A decrease in k while holding y fixed leads to an increase in the market share in Phase I (the local monopoly range). Following the intuition suggested for Proposition 2, each firm then has an incentive to decrease the rent provision in this range, which can be achieved by raising θ^* or lowering q. However, the effect on Phase II (the competition range) is different. As k decreases, competition becomes more intense. As a result, the impact of the market share effect on the firms' profits becomes relatively more important than that of the marginal effect on the firms' profit (which is further reinforced by a decrease in $\hat{\theta}$), therefore each firm has an incentive to raise rent provisions, which can be achieved by lowering θ^* or raising q. So the effects on θ^* and q of decreasing k in the two phases work in opposite directions. The net effect depends on which effect dominates.[24] When $k \in (0, k^*)$, i.e., when the initial competition between the two firms is not too weak, the competition range is more important relative to the local monopoly range,[25] thus the effect in the competition range dominates and more consumer types are covered by each firm and quality distortions decrease in equilibrium. On the other hand, when $k \in \left(k^*, \frac{4}{3}\right)$, i.e., when the initial competition between the two firms is weak, the local monopoly range is relatively more important,[26] thus the effect in the local monopoly range dominates and fewer consumer types are covered by each firm, though the effect on quality distortions is not uniform: as k decreases, there is a cutoff type, say $\widetilde{\theta}$, such that when $\theta \in [0, \widetilde{\theta})$, q moves downward, while when $\theta \in (\widetilde{\theta}, 1)$, q moves slightly upward. This non-uniform effect makes perfect sense. When $k \in \left(k^*, \frac{4}{3}\right)$, competition is weak so the movement of the quality schedule should follow the pattern in the monopoly case. This explains why as k decreases the quality schedule in the lower type range moves downward while the schedule in the higher type range remains almost unchanged—recall that in the monopoly case, as k decreases the schedule q in the partial coverage range moves downward, while it stays the same in the full coverage range.

Again our computations show that the effect of changing k on either θ^{*D} or quality distortions over the range $k > k^*$ is not quantitatively significant. However, it is qualitatively important as it provides a "continuity" for our intuitions to work when moving from monopoly to duopoly.

5. Extension to n firms

In this section we extend our analysis to any arbitrary finite number n of firms. Specifically, in the horizontal dimension there are n brands owned and operated by n distinct

[24] In terms of the rent provision schedule y, a decrease in k tends to increase $y(\theta)$ in the competition range and decrease $y(\theta)$ in the local monopoly range. But y has to be continuous at the junction of two ranges to satisfy the IC constraint.

[25] In the limit as $k \to 0$, the local monopoly range disappears.

[26] When $k \geqslant \frac{4}{3}$, the competition range disappears and both firms behave as if they were local monopolists.

firms ($n \geq 2$), the locations of which evenly split the unit circle; and each firm offers vertically differentiated products. Each firm's objective is to maximize the profit from its own brand, given the other firms' menus of offers. Again we look for symmetric Bertrand–Nash equilibria in which each firm makes the same menu of offers.[27] An n-tuple $(y^*, \dots, y^*)$ constitutes a symmetric equilibrium if, given that all other firms offer $y^*(\theta)$ for $\theta \in [\theta^*, 1]$, each firm's best response is also to choose $y_i(\theta) = y^*(\theta)$, $\theta \in [\theta^*, 1]$.

Given that all firms other than firm i offer the schedule $y^*(\theta)$, $\theta \in [\theta^*, 1]$, it can be easily verified that firm i's relaxed program (ignoring the constraint of the monotonicity of q_i) is

$$\begin{aligned}
\max \int_{\theta_i^*}^{\hat\theta} & [\theta q_i(\theta) - y_i(\theta) - c(q_i(\theta))] \frac{y_i(\theta)}{k}\, d\theta \\
& + \int_{\hat\theta}^{1} [\theta q_i(\theta) - y_i(\theta) - c(q_i(\theta))] \cdot \left[\frac{1}{2n} + \frac{1}{2k}(y_i(\theta) - y^*(\theta))\right] d\theta \\
\text{subject to} \quad & y_i'(\theta) = q_i(\theta)\,, y_i(\theta_i^*) = 0,\ \theta_i^* \text{ free} \\
& y_i(\hat\theta) = \frac{k}{n} - y^*(\hat\theta),\ \hat\theta \text{ free},\ y_i(1) \text{ free.}
\end{aligned}$$

Following an analysis parallel to that in the previous section, we can demonstrate that firm i's equilibrium rent provision $y^*(\theta)$ in the local monopoly range ($\theta < \hat\theta$) is the same as that in the duopoly model which is independent of n. The equilibrium rent provision in the competition range ($\theta > \hat\theta$) and the optimal switching point $\hat\theta$ are characterized by the following system:

$$\begin{aligned}
y'' &= 2 - \frac{n}{k}(\theta y' - y - \tfrac{1}{2} y'^2) \\
y(\hat\theta) &= k/2n \\
y'(\hat\theta) &= \sqrt{3k/2n} \\
y'(1) &= 1.
\end{aligned} \tag{14}$$

If we define $k' = k/n$ as the normalized degree of horizontal differentiation, then by inspection, in terms of k' the differential equation system (14) is exactly the same as the differential equation system (13) in the duopoly case (where $k' = k/2$). This implies that the analysis of the n-firm case can be translated into the analysis of the duopoly case through normalizing k by n, and in terms of k' the solution to the n-firm model is the same as the solution to the duopoly model. Thus all the results from the duopoly model carry over to the n-firm competitive model. In particular, the n-firm competitive model has a unique symmetric equilibrium, and this equilibrium exhibits perfect sorting, hence the participation threshold θ^* becomes a measure for the market coverage of

[27] As a direct consequence each firm is effectively competing with two adjacent firms, a common feature implied by the Salop model.

each firm.[28] Moreover, the effect of an increase in n (while holding k fixed) on the equilibrium is exactly the same as the effect of a decrease in k on the duopoly equilibrium. To re-state the results in the duopoly case in terms of k', define $k^{*\prime} = k^*/2 \approx .455$. Then as k' increases, for $k' < k^{*\prime}$, θ^* increases and q decreases, for $k^{*\prime} < k' < \frac{2}{3}$, θ^* decreases while q increases for lower types but decreases for higher types, and for $k' \geq \frac{2}{3}$, both θ^* and q are independent of k'. Translating this into n-firm case, we have the following result.

Proposition 7. *Fix $k > 0$ and define $n^* = k/k^{*\prime}$. When $n > n^*$, an increase in n leads to more consumer types being served by each firm and smaller quality distortions; when $n \in (1.5k, n^*)$, an increase in n leads to fewer consumer types being served by each firm and larger quality distortions for lower types and smaller quality distortions for higher types; when $n \leq 1.5k$, each firm is a local monopolist, hence the market coverage and quality distortions are independent of n.*

This result implies that the effect of increasing competition on market coverage or quality distortions depends on the initial state of competition, and that the effect is not monotonic.

Our two-brand monopoly can be extended to an$n$-brand multi-product monopoly by a similar normalization. Thus Proposition 2 can be extended to imply that as a monopolist offers more brands, fewer consumer types are covered by each brand. So for a multi-product monopolist, horizontal brand variety and vertical market coverage are substitutes.

6. Discussion

One main restriction in our preceding analysis is that we assume uniform distributions for consumer types. While maintaining this assumption is mainly for ease of equilibrium analysis, it is not entirely clear whether our main results hold also for other distributions. We now address this robustness issue.

Suppose consumer (vertical) types are distributed according to a CDF F over $[0,1]$ with density function f, where $f(\theta) > 0$ for all $\theta \in [0,1]$.[29] Following derivations similar to those in Section 3, it can be verified that under monopoly, Phase I (partial coverage range) is characterized by the differential equation

$$3y - \tfrac{1}{2}y'^2 - yy'' + \frac{f'}{f}y(\theta - y') = 0 \tag{15}$$

[28]In Gal-Or's (1983) quantity-setting model, symmetric Cournot equilibria may exist when the number of firms is small, but may fail to exist as the number of firms becomes larger. In contrast, in our model a symmetric Bertand–Nash equilibrium always exists and is unique.

[29]We continue to assume that consumers' horizontal types are uniformly distributed for two reasons. First, this is standard in the Hotelling–Salop model. Second, a non-uniform distribution necessarily leads to asymmetric equilibria, which are too difficult to characterize. Note that Rochet and Stole (2002) allow for a general distribution for horizontal types because their focus is on consumers' random participation.

with endpoint conditions $y(\theta^*) = 0$ and $y(\hat{\theta}) = k/4$. Similarly, Phase II (full coverage range) is characterized by

$$y'' = 2 + \frac{f'}{f}(\theta - y')$$

which can be further reduced to[30]

$$y' = \theta - \frac{1 - F(\theta)}{f(\theta)}.$$

Thus for $q(\theta)$ to be strictly increasing (perfect sorting) over $[\hat{\theta}, 1]$, a sufficient condition is that the hazard rate function of $F(\theta)$ be increasing.

Now following derivations similar to those in Section 4, it can be verified that under duopoly, Phase I (local monopoly range) is characterized by the differential equation

$$3y - \tfrac{1}{2}y'^2 - yy'' + \frac{f'}{f}y(\theta - y') = 0$$

with endpoint conditions $y(\theta^*) = 0$ and $y(\hat{\theta}) = k/4$ (which is the same as in the monopoly case). Phase II (competition range) is characterized by

$$y'' = 2 - \frac{2}{k}\left(\theta y' - y - \tfrac{1}{2}y'^2\right) + \frac{f'}{f}(\theta - y').$$

By working with some specific distribution functions (e.g. the truncated exponential or generalized uniform distributions), it is clear that there is no analytical solution to either the monopoly or the duopoly differential equation system. Thus an analytical solution is generally unavailable for general distribution functions. The reduced order technique introduced in the proof of Proposition 3 (and in Rochet and Stole) cannot be applied to simplify the differential equations either. We thus turn to numerical computations to characterize the equilibrium given specific distributions.

We first work with the case in which $f(\theta) = e^{\theta}/(e-1)$ for $\theta \in [0,1]$ (a truncated exponential distribution). Our computation shows that the equilibrium exhibits perfect sorting under both monopoly and duopoly. The schedule $q(\theta)$ for the case $k = 0.4$ and the whole schedule $\theta^*(k)$ under both monopoly and duopoly are depicted in Figure 5.

As can be seen from the figure, θ^{*M} is always decreasing as k increases. But for the duopoly model, there is a cutoff k^* such that for $k \in (0, k^*)$, θ^{*D} is increasing in k, and for $k \in \left(k^*, \frac{4}{3}\right)$, θ^{*D} is decreasing in k (for $k \geq \frac{4}{3}$, $\theta^{*D} = \theta^{*M}$ is independent of k). Our computation shows that the turning point k^* is approximately 0.90. The decreasing trend of θ^{*D} in the range $\left(k^*, \frac{4}{3}\right)$ is not quantitatively significant; however, the increasing trend of θ^{*D} in the range $(0, k^*)$ is quantitatively significant. This pattern is very similar to the one derived for the uniform distribution case (Figure 4). So all the results demonstrated from the uniform distribution also carry over to this (truncated) exponential distribution case. We also examine a generalized uniform distribution $f(\theta) = 2\theta$ for $\theta \in [0,1]$.

[30] Define $K \equiv f(\theta)(\theta - y') + \int_0^{\theta} f(s)\,ds$. In light of (15) it can be verified that $K' = 0$. The condition $y'(1) = 1$ then implies that $K = 1$.

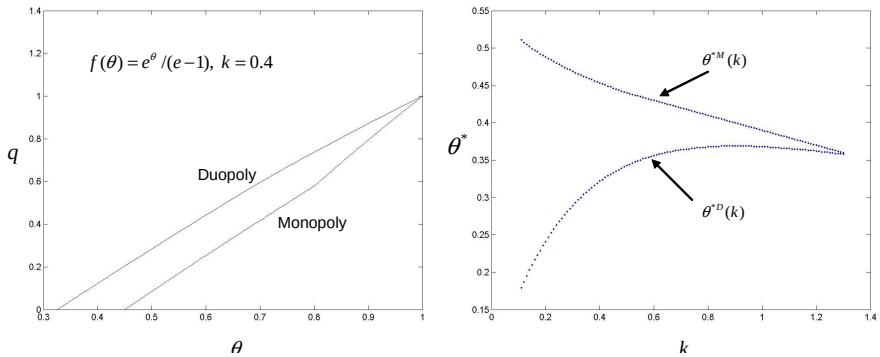

FIGURE 5. The exponential distribution case.

Again our computation shows that the equilibrium exhibits perfect sorting under both monopoly and duopoly. The schedule $q(\theta)$ for the case $k = 0.3$ and the whole schedule $\theta^*(k)$ under both monopoly and duopoly are depicted in Figure 6.

The comparison is, once again, qualitatively not different from the case of the uniform distribution. In fact, for all the cases (with increasing hazard rate functions) that we have computed, the comparisons between the schedules $\theta^{*D}(k)$ and $\theta^{*M}(k)$ are qualitatively the same as those obtained in the uniform distribution case. We thus believe that the results derived from our main model are fairly robust, and our focus on the uniform distribution is primarily for ease of equilibrium characterization.

7. Conclusion

In this paper we extend the analysis of Rochet and Stole (1997, 2002) by considering partial coverage of consumer types on the vertical dimension in a market with both vertically and horizontally differentiated products. In each market structure that we analyze, the equilibrium exhibits perfect sorting (bunching never occurs), and the quality distortion is maximal for the lowest type. Our results are thus quite different from those obtained by Rochet and Stole (1997, 2002).

By focusing on the case where the lowest type of consumers being served is endogenously determined, we are able to study also the effect of varying horizontal differentiation (competition) on vertical market coverage and quality distortions. When moving from monopoly to duopoly, more consumer types are covered by each brand (firm), and the quality distortions become smaller. As the market structure becomes more competitive, the effect of increased competition exhibits some non-monotonic features: when the initial competition is not too weak, a further increase in the number of firms leads to more types of consumers being served and a reduction in quality distortions; when the initial competition is weak, an increase in the number of firms leads to fewer types of consumers being served, though the effect on quality distortions is not uniform.

For tractability reasons we assume uniform distributions for consumer types in our main analysis. However, the driving force behind our results, i.e., the interaction

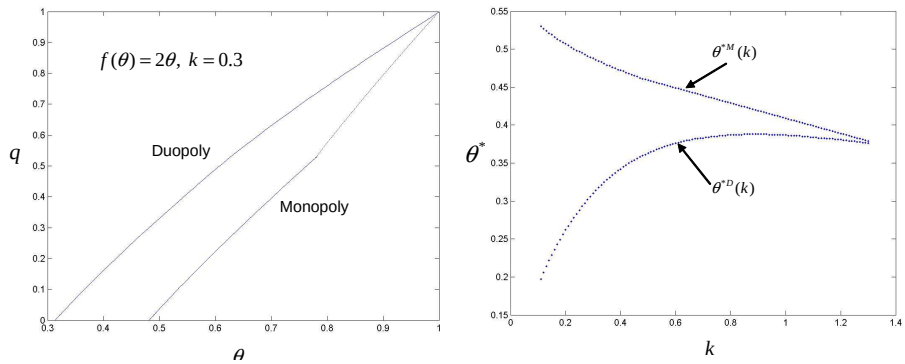

FIGURE 6. The generalized uniform distribution case.

between horizontal differentiation (competition) and screening on the vertical dimension, is fairly robust and is not restricted to specific distributions. Our results about the effect of competition on market coverage and quality distortions have testable implications, which are left for future research.

APPENDIX

PROOF OF PROPOSITION 3. Following the derivations preceding the proposition, the proof is completed by showing that for all $k \in \left(0, \frac{4}{3}\right]$ there is a unique $\hat{\theta} \in (0,1]$ and a unique $y(\theta)$ defined over $[\hat{\theta}, 1]$ satisfying the differential equation system (13). Moreover, the solution of $y(\theta)$ is strictly convex.

First letting $z(\theta) = y(\theta) - \frac{1}{2}\theta^2$, we have

$$z''(\theta) = 1 + \frac{1}{k}(z'^2(\theta) + 2z(\theta)). \tag{16}$$

Let $z'(\theta) = v(z(\theta))$. Then $z''(\theta) = v'(z)z'(\theta) = v\,v'(z)$. Equation (16) thus becomes

$$v\frac{dv}{dz} = 1 + \frac{1}{k}(v^2 + 2z). \tag{17}$$

Substituting $w(z) = v^2(z)$ into (17), we have $w' - 2w/k = 2 + 4z/k$, which leads to

$$w(z) = c\,e^{2z/k} - 2z - 2k,$$

where c is a parameter to be determined by the boundary conditions.

The system (13) can now be written in terms of the function z as follows:

$$\begin{aligned} (z'(\theta))^2 &= c\,e^{2z(\theta)/k} - 2z(\theta) - 2k \\ z(\hat{\theta}) &= \tfrac{1}{4}k - \tfrac{1}{2}\hat{\theta}^2 := \hat{z} \\ z'(\hat{\theta}) &= \tfrac{1}{2}\sqrt{3k} - \hat{\theta} \\ z'(1) &= 0. \end{aligned} \tag{18}$$

Define α such that $c = k\alpha e^{-2\hat{z}/k}$ and δ such that $\hat{\theta} = \frac{1}{2}\sqrt{3k}\delta$ ($\hat{\theta} \in (0,1]$ implies $\delta \in (0, 2/\sqrt{3k})$). Define also $u(\theta) = 2(z(\theta) - \hat{z})/k$. Then we have

$$u'^2 = \frac{4}{k^2} z'^2 = \frac{4}{k^2}(k\alpha e^u - 2z - 2k) = \frac{4}{k}\left(\alpha e^u - u - \frac{2}{k}\hat{z} - 2\right).$$

Letting $f(u) = \alpha(e^u - 1) - u + \beta$, where $\beta = \alpha - \frac{2}{k}\hat{z} - 2$, we have $u'^2 = 4f(u)/k$.

At $\hat{\theta}$, $u(\hat{\theta}) = 0$, $u'(\hat{\theta}) = \sqrt{3/k}(1-\delta)$, hence

$$\beta = \frac{k}{4}u'^2(\hat{\theta}) = \frac{3}{4}(1-\delta)^2,\ \alpha = \beta + \frac{2}{k}\hat{z} + 2 = \tfrac{13}{4} - \tfrac{3}{2}\delta.$$

The system (18) can now be rewritten as follows:

$$u'^2 = \frac{4}{k} f(u) \tag{19}$$

$$u(\hat{\theta}) = 0 := \hat{u} \tag{20}$$

$$u'(\hat{\theta}) = \sqrt{3/k}(1-\delta) := \hat{u}' \tag{21}$$

$$u'(1) = 0 := u_1' \tag{22}$$

where

$$f(u) = \alpha(e^u - 1) - u + \beta = \left(\tfrac{13}{4} - \tfrac{3}{2}\delta\right)(e^u - 1) - u + \tfrac{3}{4}(1-\delta)^2.$$

For notational convenience let $u_1 =: u(1)$. Then $u_1' = 0 \Rightarrow f(u_1) = 0$.

From (21)–(22) it can be verified that $\hat{\theta} = 1 \Rightarrow k = \frac{4}{3}$. So for $k \in \left(0, \frac{4}{3}\right)$ we must have $\hat{\theta} < 1$, or $\delta < 2/\sqrt{3k}$.

The rest of the proof consists of six steps.

Step 1. *Equation (19) implies $u' = -(2/\sqrt{k})\sqrt{f(u)}$ and $\delta \geqslant 1$.*

Proof. Suppose not. Then $u' = (2/\sqrt{k})\sqrt{f(u)} \geq 0$. By (21), $\delta \leq 1$ and $\alpha \geq \frac{7}{4}$, which imply $f'(u) = \alpha e^u - 1 \geq \frac{7}{4} - 1 > 0$ for all $u \geq 0$. But then $f(u_1) > f(\hat{u}) = f(0) = \beta \geq 0$, a contradiction. Therefore we must have $u' = -(2/\sqrt{k})\sqrt{f(u)} \leq 0$ and hence $\delta \geqslant 1$. Since u is decreasing, we have $u_1 \leq \hat{u} = 0$. It can be verified that for $k \in \left(0, \frac{4}{3}\right)$, $u_1 = \hat{u} = 0$ is impossible.[31] Hence $u_1 < \hat{u} = 0$ for $k \in \left(0, \frac{4}{3}\right)$, and $f(u) \geq 0$ on $[u_1, 0]$. ◁

Step 2. *In the solution to the system (19)–(22), $\alpha > 0$, which implies that the original solution y is strictly convex.*

Proof. Suppose not, i.e., suppose $\alpha \leq 0$. Then $f'(u) = \alpha e^u - 1 < 0$, which implies that $f(\hat{u}) < f(u_1) = 0$. But $f(\hat{u}) = \alpha(e^{\hat{u}} - 1) - \hat{u} + \beta = \beta \geqslant 0$, contradiction. So $\alpha > 0$.

Since

$$y'' = 1 + z'' = \frac{1}{k} c e^{2z/k} = \alpha e^{2(z-\hat{z})/k},$$

$\alpha > 0$ (or $\delta < \frac{13}{6}$) implies that the original solution y is strictly convex. ◁

[31] We have $u_1 = \hat{u} \Rightarrow u = 0$, which implies $z = \hat{z}$ and $\hat{\theta} = \frac{1}{2}\sqrt{3k}$. Therefore $y(\theta) = \frac{1}{2}\theta^2 + \hat{z} = \frac{1}{2}\theta^2 + \frac{k}{4} - \frac{1}{2}\hat{\theta}^2 = \frac{1}{2}\theta^2 - \frac{1}{8}k$. But then $y(\theta)$ does not satisfy the differential equation in system (13), a contradiction.

STEP 3. *Given δ (or $\hat{\theta}$), a solution for u (and hence y) exists and is unique.*

PROOF. Since $f''(u) = \alpha e^u > 0$, f is strictly convex (with $f(\pm\infty) = \infty$). Hence $f(u) > 0$ on $(u_1, 0]$. We have $f'(u) = \alpha e^u - 1 = 0 \Rightarrow u_{\min} = -\ln\alpha$. Let $A(\delta) := \min f(u) = f(-\ln\alpha) = \ln\alpha + \frac{3}{4}(\delta^2 - 2)$. Since $f(u_1) = 0$, we must have $A(\delta) \le 0$.

We next show that $A(\delta) < 0$. Suppose not. Then $u_1 = u_{\min} = -\ln\alpha < 0$, which implies $f(u) \approx a(u - u_1)^2$ near u_1, where a is a positive real number. The condition $du/\sqrt{f(u)} = -(2/\sqrt{k})d\theta$ implies that

$$\int_{u_1}^{0} \frac{du}{\sqrt{f(u)}} = -\frac{2}{\sqrt{k}} \int_{1}^{\hat{\theta}} d\theta = \frac{2}{\sqrt{k}}(1 - \hat{\theta}) < \infty. \tag{23}$$

But on the other hand,

$$\int_{u_1}^{0} \frac{du}{\sqrt{f(u)}} = \frac{1}{\sqrt{a}} \int_{u_1}^{\hat{u}} \frac{du}{u - u_1} = \infty,$$

a contradiction. Therefore $A(\delta) < 0$ and hence in the neighborhood of u_1, $f(u) = O(u - u_1)$.

Define

$$\Phi(u) := \int_{0}^{u} \frac{dv}{\sqrt{f(v)}} = \int_{\hat{\theta}}^{\theta} -\frac{2}{\sqrt{k}}\, ds = -\frac{2}{\sqrt{k}}(\theta - \hat{\theta}).$$

Note that $\Phi(u)$ is well defined for any $u \in [u_1, 0]$, as $f(u) = O(u - u_1)$ near u_1 (which implies $\left|\int_0^{u_1} dv/\sqrt{f(v)}\right| < \infty$).

Since $\Phi(u)$ is a strictly increasing function over $[u_1, 0]$, inverting we have

$$u(\theta) = \Phi^{-1}\left(-\frac{2}{\sqrt{k}}(\theta - \hat{\theta})\right) \text{ for } \theta \in [\hat{\theta}, 1]. \tag{24}$$

Thus given $\hat{\theta}$, u (and hence y) is uniquely determined by (24). ◁

It remains to show that $\hat{\theta}$ (or δ) exists and is unique.

STEP 4. *In the solution, $\delta \in [1, \min\{\delta_0, 2/\sqrt{3k}\})$ (where δ_0 is defined below).*

PROOF. Since $-\ln\alpha = u_{\min} < u_1 < 0$, we have $\alpha > 1$ or $\delta < \frac{3}{2}$. We thus have $\delta \in [1, \frac{3}{2})$ (from Step 1).

It is straightforward to verify that $A(\delta)$ is strictly increasing over the interval $\left[1, \frac{3}{2}\right)$ and there is a unique $\delta_0 \in \left[1, \frac{3}{2}\right)$ such that $A(\delta_0) = 0$. Since $A(\delta) < 0$, we thus have $\delta \in [1, \delta_0)$. Combining this with $\delta < 2/\sqrt{3k}$, in the solution to the system (19)–(22) we must have $\delta \in [1, \min\{\delta_0, 2/\sqrt{3k}\})$. ◁

By (23) we have $\int_{u_1}^{0} du/\sqrt{f(u)} = (2/\sqrt{k})(1 - \hat{\theta}) = (2/\sqrt{k}) - \sqrt{3}\delta$. Define

$$\xi(\delta) = \sqrt{3}\delta + \int_{u_1}^{0} \frac{du}{\sqrt{f(u)}}. \tag{25}$$

Step 5. *Given any* $k \in \left(0, \frac{4}{3}\right)$, *there exists* $\delta \in (1, \min\{\delta_0, 2/\sqrt{3k}\})$ *satisfying* $\xi(\delta) = 2/\sqrt{k}$.

Proof. First, $f - \beta + u + \alpha = \alpha e^u$ implies $(f - \beta + u + \alpha)' = f - \beta + u + \alpha$. That is, $f' - f - u = \text{constant} = f'(u_1) - f(u_1) - u_1 = f'(u_1) - u_1$. Hence $f' - f - (u - u_1) = f'(u_1) > 0$, which leads to

$$f'(u_1)\int_{u_1}^{0} \frac{1}{\sqrt{f}}\,du = \int_{u_1}^{0} \frac{f'}{\sqrt{f}}\,du - \int_{u_1}^{0} \sqrt{f}\,du - \int_{u_1}^{0} \frac{u - u_1}{\sqrt{f}}\,du. \tag{26}$$

Define

$$\xi_1(\delta) = \int_{u_1}^{0} \sqrt{f}\,du \quad \text{and} \quad \xi_2(\delta) = \int_{u_1}^{0} \frac{u - u_1}{\sqrt{f}}\,du.$$

Note that $f'(u_1) = \alpha e^{u_1} - 1 > 0$ and $\int_{u_1}^{0} (f'/\sqrt{f})\,du = 2\sqrt{f(0)} = 2\sqrt{\beta} = \sqrt{3}(\delta - 1)$. Therefore by (26) we have

$$\xi(\delta) = \frac{1}{\alpha e^{u_1} - 1}\left[\sqrt{3}(\delta - 1) - \xi_1(\delta) - \xi_2(\delta)\right] + \sqrt{3}\delta. \tag{27}$$

Since $u_1(\delta)$ is continuous in δ, both $\xi_1(\delta)$ and $\xi_2(\delta)$ are also continuous in δ. Therefore, $\xi(\delta)$ is continuous in δ.

First, consider $\delta \to 1^+$. It is easily verified that $\beta \to 0^+$ and $\alpha \to (\frac{7}{4})^-$. Hence $f(u) \to g(u) = \frac{7}{4}(e^u - 1) - u$, and $u_1(\delta) \to 0^-$. By (27), $\xi(\delta) < (1/(\alpha e^{u_1} - 1))\sqrt{3}(\delta - 1) + \sqrt{3}\delta \to \sqrt{3}$. Since $\sqrt{3} < 2/\sqrt{k}$, we have $\xi(\delta) < 2/\sqrt{k}$ for δ sufficiently close to 1^+.

Second, consider $\delta \to b = \min\{\delta_0, 2/\sqrt{3k}\}$ from the left. We discuss the following two cases.

$\delta_0 > 2/\sqrt{3k}$: Then when $\delta \to b^- = (2/\sqrt{3k})^-$, $\xi(\delta) > \sqrt{3}\delta = \sqrt{3}b = 2/\sqrt{k}$ (the inequality is due to (25)).

$\delta_0 \le 2/\sqrt{3k}$: For $\delta \to b^- = \delta_0^-$, $A(\delta) = f_{\min} \to 0^-$ (since $A(\delta_0) = 0$). So $u_1 \to (-\ln\alpha)^+$, and by (25), $\xi(\delta) \to \infty$. So when $\delta \to b^-$, $\xi(\delta) > 2/\sqrt{k}$.

By the mean-value theorem, there exists $\delta \in (1, \min\{\delta_0, 2/\sqrt{3k}\})$ such that $\xi(\delta) = 2/\sqrt{k}$. ◁

Step 6. *The solution in Step 5 is unique.*

Proof. We have

$$f(u_1) = \alpha(e^{u_1} - 1) - u_1 + \beta = 0. \tag{28}$$

Differentiating (28) with respect to δ, we have

$$-\tfrac{3}{2}(e^{u_1} - 1) + \tfrac{3}{2}(\delta - 1) + (\alpha e^{u_1} - 1)u_1' = 0,$$

which gives

$$u_1' = \frac{\frac{3}{2}(e^{u_1} - \delta)}{\alpha e^{u_1} - 1} = \frac{1}{\eta}\tfrac{3}{2}(e^{u_1} - \delta),$$

where $\eta = \alpha e^{u_1} - 1 > 0$.

By (27),

$$\xi_1' = \frac{d\xi_1}{d\delta} = \frac{d\xi_1}{du_1}\frac{du_1}{d\delta} = 0 \cdot \frac{du_1}{d\delta} = 0$$
$$\xi_2' = \frac{d\xi_2}{du_1}u_1' = \left(0 + \int_{u_1}^{0} \frac{-1}{\sqrt{f}}\,du\right)u_1' = -(\xi - \sqrt{3}\delta)u_1',$$

so

$$\begin{aligned}
\xi' &= \sqrt{3} + \frac{1}{\eta}(\sqrt{3} - \xi_1' - \xi_2') - \frac{1}{\eta^2}\left(\alpha e^{u_1}u_1' - \tfrac{3}{2}e^{u_1}\right)[\sqrt{3}(\delta - 1) - \xi_1 - \xi_2] \\
&= \sqrt{3} + \frac{1}{\eta}[\sqrt{3} + (\xi_1 - \sqrt{3}\delta)u_1'] + \frac{e^{u_1}(\frac{3}{2} - \alpha u_1')}{\eta}(\xi - \sqrt{3}\delta) \\
&= \sqrt{3} + \frac{1}{\eta}\sqrt{3} + \frac{1}{\eta}(\xi - \sqrt{3}\delta)\left[u_1'(1 - \alpha e^{u_1}) + \tfrac{3}{2}e^{u_1}\right] \\
&= \sqrt{3} + \frac{1}{\eta}\sqrt{3} + \frac{1}{\eta}(\xi - \sqrt{3}\delta)\tfrac{3}{2}\delta \\
&> 0 \ \ (\text{since } \xi - \sqrt{3}\delta = \int_{u_1}^{0} \frac{du}{\sqrt{f(u)}} > 0).
\end{aligned}$$

Therefore $\xi(\delta)$ is strictly increasing in $\delta \in (1, \min\{\delta_0, 2/\sqrt{3k}\})$, which implies that there is a unique δ satisfying $\xi(\delta) = 2/\sqrt{k}$. ◁

This completes the proof. □

Proof of Proposition 4. Suppose $\theta^{*M} \le \theta^{*D}$. Since $\hat{\theta}^D - \theta^{*D} = \hat{\theta}^M - \theta^{*M} = \sqrt{k/3}$, $\hat{\theta}^M \le \hat{\theta}^D$. By the quality provision schedules in the partial coverage range we have

$$q_M(\hat{\theta}^M) = q_D(\hat{\theta}^D) = \sqrt{3k}/2.$$

From the quality provision schedule in the full coverage range under monopoly,

$$\begin{aligned}
&q_M'(\theta) = 2 > 0 \text{ for } \theta \in [\hat{\theta}^M, \hat{\theta}^D] \\
&\quad \Rightarrow q_M(\hat{\theta}^D) \geqslant q_D(\hat{\theta}^D).
\end{aligned} \tag{29}$$

From (12),

$$q_D'(\theta) = 2 - \frac{2}{k}\left[\theta y' - y - \tfrac{1}{2}y'^2\right].$$

In equilibrium, a firm's profit from a type θ consumer is positive for $\theta > \theta^{*D}$, i.e. $\theta y' - y - \frac{1}{2}y'^2 > 0$, hence

$$q_M'(\theta) = 2 > q_D'(\theta) \text{ for } \theta \in [\hat{\theta}^D, 1]. \tag{30}$$

Note that $k \in \left(0, \frac{4}{3}\right)$ implies $\hat{\theta}^D < 1$. Combining this with (29) and (30) we have $q_M(1) > q_D(1)$, which contradicts the fact that $q_M(1) = q_D(1) = 1$. Therefore $\theta^{*M} > \theta^{*D}$ in equilibrium.

To show that $q_D(\theta) > q_M(\theta)$, we consider the following cases.

- For $\theta \in [\theta^{*D}, \hat{\theta}^D]$, by the quality provision schedules in the partial coverage range we have $q_D(\theta) > q_M(\theta)$ as $\theta^{*D} < \theta^{*M}$.
- For $\theta \in (\hat{\theta}^D, \hat{\theta}^M]$, $q_D(\theta) > q_D(\hat{\theta}^D) = \sqrt{3k}/2$ and $q_M(\theta) \le q_M(\hat{\theta}^M) = \sqrt{3k}/2$. Hence $q_D(\theta) > q_M(\theta)$.
- For $\theta \in (\hat{\theta}^M, 1)$, $q_D'(\theta) < 2 = q_M'(\theta)$ and $q_M(1) = q_D(1)$ imply that $q_D(\theta) > q_M(\theta)$.

To sum up, $q_D(\theta) > q_M(\theta)$, hence $\theta - q_D(\theta) < \theta - q_M(\theta)$ for $\theta \in [\theta^{*D}, 1)$, which implies that quality distortion is smaller in the duopoly case. □

Proof of Proposition 5. The following lemma is needed in the proof.

Lemma 2. *Suppose A, B, C, D, C' and D' are all strictly positive, with $D' > D$ and $A/B < C/D < C'/D'$. Then $(A+C)/(B+D) < (A+C')/(B+D')$.*

Proof. We first show that for $\lambda > 0$, $(A+\lambda C)/(B+\lambda D)$ is strictly increasing in λ. To verify this, taking the derivative with respect to λ,

$$\begin{aligned}\left(\frac{A+\lambda C}{B+\lambda D}\right)' &\propto C(B+\lambda D) - D(A+\lambda C)\\ &= CB - DA > 0,\end{aligned}$$

where the last inequality follows from $A/B < C/D$. Applying the above property, we have

$$\frac{A+C}{B+D} < \frac{A+C(D'/D)}{B+D(D'/D)} = \frac{A+C(D'/D)}{B+D'} < \frac{A+C'}{B+D'},$$

where the first inequality follows from $D' > D$ and the second inequality follows from $C/D < C'/D'$. □

Let Eq_D and Eq_M be the average quality of products offered under duopoly and monopoly, respectively. Specifically,

$$Eq_D = \frac{\int_{\theta^{*D}}^{\hat{\theta}^D} q_D(\theta)\frac{y_D(\theta)}{k}\,d\theta + \frac{1}{4}\int_{\hat{\theta}^D}^{1} q_D(\theta)\,d\theta}{\int_{\theta^{*D}}^{\hat{\theta}^D} \frac{y_D(\theta)}{k}\,d\theta + \frac{1}{4}(1-\hat{\theta}^D)}$$

$$Eq_M = \frac{\int_{\theta^{*M}}^{\hat{\theta}^M} q_M(\theta)\frac{y_M(\theta)}{k}\,d\theta + \frac{1}{4}\int_{\hat{\theta}^M}^{1} q_M(\theta)\,d\theta}{\int_{\theta^{*M}}^{\hat{\theta}^M} \frac{y_M(\theta)}{k}\,d\theta + \frac{1}{4}(1-\hat{\theta}^M)}.$$

From the previous analysis, we have

$$\int_{\theta^{*D}}^{\hat{\theta}^D} q_D(\theta)\frac{y_D(\theta)}{k}\,d\theta = \int_{\theta^{*M}}^{\hat{\theta}^M} q_M(\theta)\frac{y_M(\theta)}{k}\,d\theta$$

$$\int_{\theta^{*D}}^{\hat{\theta}^D} \frac{y_D(\theta)}{k}\,d\theta = \int_{\theta^{*M}}^{\hat{\theta}^M} \frac{y_M(\theta)}{k}\,d\theta.$$

Since $\hat{\theta}^D < \hat{\theta}^M$ and $q_D(\theta) > q_M(\theta)$ for any $\theta \in (\hat{\theta}^M, 1)$, we also have

$$\tfrac{1}{4}(1-\hat{\theta}^M) < \tfrac{1}{4}(1-\hat{\theta}^D) \text{ and } \tfrac{1}{4}\int_{\hat{\theta}^D}^1 q_D(\theta)\,d\theta > \tfrac{1}{4}\int_{\hat{\theta}^M}^1 q_M(\theta)\,d\theta.$$

Moreover,

$$\frac{\int_{\theta^{*M}}^{\hat{\theta}^M} q_M(\theta)\frac{y_M(\theta)}{k}\,d\theta}{\int_{\theta^{*M}}^{\hat{\theta}^M} \frac{y_M(\theta)}{k}\,d\theta} < q_M\left(\hat{\theta}^M\right) < \frac{\frac{1}{4}\int_{\hat{\theta}^M}^1 q_M(\theta)\,d\theta}{\frac{1}{4}(1-\hat{\theta}^M)},$$

since $q_M(\theta)$ is strictly increasing. Thus in view of Lemma 2, to prove $Eq_D > Eq_M$ it is sufficient to show that

$$\frac{\int_{\hat{\theta}^D}^1 q_D(\theta)\,d\theta}{(1-\hat{\theta}^D)} > \frac{\int_{\hat{\theta}^M}^1 q_M(\theta)\,d\theta}{(1-\hat{\theta}^M)}. \tag{31}$$

Note that $q_D(\hat{\theta}^D) = q_M(\hat{\theta}^M)$, $q_D(1) = q_M(1)$, and $q_M(\theta)$ is linear. Let $q_L(\theta)$ be a linear function satisfying $q_L(\hat{\theta}^D) = q_D(\hat{\theta}^D)$ and $q_D(1) = q_L(1)$. Then

$$\frac{\int_{\hat{\theta}^D}^1 q_L(\theta)\,d\theta}{(1-\hat{\theta}^D)} = \frac{\int_{\hat{\theta}^M}^1 q_M(\theta)\,d\theta}{(1-\hat{\theta}^M)} = \tfrac{1}{2}(1+q_D(\hat{\theta}^D)).$$

Now for (31) to hold, it is sufficient to show that

$$\int_{\hat{\theta}^D}^1 q_D(\theta)\,d\theta > \int_{\hat{\theta}^D}^1 q_L(\theta)\,d\theta. \tag{32}$$

Note that $q_D(\theta)$ is concave for $\theta \in (\hat{\theta}^D, 1)$. To see this, differentiating (12) with respect to θ, we have

$$q'' = y''' = -\frac{2}{k}(\theta - y')y'' < 0$$

since $y'' > 0$ and $\theta - y' > 0$ for $\theta \in (\hat{\theta}^D, 1)$. The concavity of $q_D(\theta)$ implies that $q_D(\theta) > q_L(\theta)$ for any $\theta \in (\hat{\theta}^D, 1)$. Therefore, the inequality (32) holds. □

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