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Strategy-proofness and single-crossing

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This paper analyzes strategy-proof collective choice rules when individuals have single-crossing preferences on a finite and ordered set of social alternatives. It shows that a social choice rule is anonymous, unanimous, and strategy-proof on a maximal single-crossing domain if and only if it is an extended median rule with $n-1$ fixed ballots distributed over the individuals' most preferred alternatives. As a by-product, the paper also proves that strategy-proofness implies the tops-only property. It also offers a strategic foundation for the so-called "single-crossing version" of the Median Voter Theorem, by showing that the median ideal point can be implemented in dominant strategies by a direct mechanism in which every individual reveals his true preferences.

Keywords. Single-crossing, strategy-proofness, tops-only, peak rules.
JEL classification. C72, D71, D78.

1. Introduction

In social choice theory, a collective decision making process is usually represented by a social choice rule. A social choice rule associates a unique alternative from the set of feasible alternatives to every possible list of preferences of the individuals in the society. A social choice rule is said to be strategy-proof if no individual can ever benefit from misrepresenting his true preferences. A fundamental result in social choice theory, known as the Gibbard (1973)–Satterthwaite (1975) Theorem, shows that, if the set of alternatives contains at least three possible outcomes and individual preferences are not restricted in any particular way, then every strategy-proof social choice rule is dictatorial. That is, there is an individual whose preferences always dictate the final choice regardless of the other individuals' preferences.

The Gibbard–Satterthwaite Theorem holds under the so called *universal domain* assumption, which means that every profile of complete and transitive preference relations is an admissible element of the domain of the social choice rule. In many economic and political applications, however, preferences satisfy additional properties. A case in point is the single-peaked property.

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A set of preference relations is single-peaked if there is a linear order of the alternatives such that every preference relation has a unique most preferred alternative (or ideal point) over this ordering, and the preference for any other alternative monotonically decreases by moving away from the ideal point. Single-peaked preferences naturally arise in economics when a strictly quasi-concave utility function is maximized on a linear budget set. They were first proposed by Black (1948) to assure the existence of a Condorcet winner (i.e., an alternative that beats every other alternative in a sequence of pair-wise majority contests). And they represent a simple case where the conclusion of the Gibbard–Satterthwaite Theorem does not apply.

To be more specific, consider the family of efficient extended median rules, which are social choice rules that associate to each preference profile the median alternative from a list consisting of the n ideal points of the individuals and $n-1$ other alternatives from the feasible set of alternatives. An important member of this family is the well-known median choice rule, which assigns the median ideal point to every profile of individual preferences.[1] These rules are obviously non-dictatorial. In fact, they are anonymous, because the names of the individuals play no role in making social choices. They are also unanimous, in the sense that they respect any unanimous consensus in the society about the most preferred alternative. Furthermore, if individual preferences are single-peaked, then Moulin (1980) has shown that every member of this family is strategy-proof. Conversely, every anonymous, unanimous, and strategy-proof social choice rule on the domain of single-peaked preferences is an efficient extended median rule.

Although single-peakedness is an intuitive domain restriction, there are interesting problems in political economy and public economics, such as majority voting over distortionary tax rates, where individual preferences do not exhibit the single-peaked property (see, for example, Romer 1975, p. 181, and Austen-Smith and Banks 1999, pp. 114–115). In some of these cases, however, preferences do satisfy an alternative restriction called the single-crossing property. This property appears, for example, in models of income taxation and redistribution (Roberts 1977, Meltzer and Richard 1981), local public goods and stratification (Westhoff 1977, Epple and Platt 1998, Epple et al. 2001), coalition formation (Demange 1994, Kung 2006) and, more recently, in models that study the selection of policies in the market for higher education (Epple et al. 2006) and the choice of constitutional and voting rules (Barberà and Jackson 2004).

Unlike single-peakedness, the single-crossing property does not impose *a priori* any restriction on the shape of each individual preference relation. So, for example, it does not exclude preferences that do not monotonically decrease on both sides of the ideal point. That is the reason why it accommodates non-convexities that arise in some applications of majority voting. If preferences are strict orderings, what the single-crossing property requires is the existence of a linear order over the set of individual preferences with the property that, for every pair of alternatives x and y, whenever two preference relations P' and P'' coincide in ranking x above y, so do all preferences *in between*, so

[1] When n is even, ties at the median are broken in favor of either the largest or the smallest median peak depending upon the tie-breaking rule in place.

that the subset of preferences ranking one alternative above the other all lie to one side of those that have the reverse ranking.[2] Of course, if indifference between alternatives is permitted, then the set of preference relations for which x is indifferent to y must be located between the subsets with a strict ordering of these two alternatives.

In several applications, notably in models of redistribution financed by income taxation, the single-crossing property is implied by more fundamental assumptions about preferences and technologies. For instance, it holds when the heterogeneity among individuals is created by a one-dimensional parameter θ (which can be interpreted as income, productivity, an elasticity of substitution, a discount factor, etc.), and the utility over social alternatives exhibits increasing differences in θ (Milgrom and Shannon 1994). In addition, under differentiability and some mild conditions on indifference curves, the single-crossing property is also equivalent to the more familiar Spence–Mirrlees condition of incentive theory and information economics, which requires that the marginal rate of substitution be increasing in θ (Milgrom and Shannon 1994).

The single-crossing property has in many cases a substantive interpretation. A working example is the collective choice of an income tax rate. Suppose a moderately rich individual prefers a high tax rate to another relatively smaller tax rate, so that he reveals a preference for a greater redistribution of income. Then, the single-crossing property requires that a relatively poorer individual, who receives a higher benefit from redistribution, also prefers the higher tax rate. Sometimes this is interpreted in the literature by saying that there is a complementarity between income and taxation, in the sense that lower incomes increase the incremental benefit of greater tax rates. For another example, consider a strong army that prefers a large territorial concession and a small probability of war to a small concession and a high probability of war. Then, under single-crossing, a weaker army, with a lower expected payoff from war, should also prefer the large concession (Ashworth and Mesquita 2006, pp. 217–218).

Like the single-peaked property, single-crossing also guarantees the existence of a Condorcet winner and allows a simple characterization of it. The Condorcet winner is the ideal point of the median agent, where the latter is the individual whose preference takes up the median position over the ordering of individual preferences for which the single-crossing property is satisfied.[3] This result appeared first in the seminal works of Roberts (1977) and Grandmont (1978) and, more recently, in Rothstein (1991), Gans and Smart (1996), and Austen-Smith and Banks (1999). It is referred to by Myerson (1996) as the "single-crossing version" of the Median Voter Theorem (MVT). Alternatively, due to the existence of a median individual who is decisive for every subset of alternatives, Rothstein (1991) calls it the *Representative Voter Theorem* (RVT).

The problem with the Representative Voter Theorem is that, unlike the MVT over single-peaked preferences, whose non-cooperative foundation was provided by Moulin

[2]When preferences are strict, it is also possible and convenient to derive a linear order over the set of alternatives from the order of the preference relations, by defining alternative x to be "smaller than" alternative y if and only if the preference relations for which x is preferred to y lie on the left of the relations that rank y above x.

[3]Instead, under single-peakedness, the Condorcet winner is given by the median ideal point over the ordering of the alternatives for which the single-peaked property holds.

(1980), the RVT is based on the assumption that individuals honestly reveal their preferences. A natural question is therefore how legitimate the Representative Voter Theorem is when preferences are private information and individuals can report them insincerely. This question has been recently addressed by Saporiti and Tohmé (2006). They show that the single-crossing property is sufficient to ensure the existence of social choice rules that are immune to any individual and group misrepresentation of preferences. In particular, this is true for the median choice rule.

Building on Saporiti and Tohmé (2006), this paper characterizes the family of anonymous, unanimous, and strategy-proof social choice rules on a maximal single-crossing domain.[4] This family coincides with the class of *peak rules*, which are extended median rules with $n-1$ fixed ballots distributed over the individuals' most preferred alternatives. This class includes the median choice rule as a particular case. Hence, the main message of the analysis is that the single-crossing property is another meaningful domain restriction where majority voting works with "maximal" incentive properties. The paper explains the source of this good property of single-crossing domains, and how far we can go in changing majority rule.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 contains the main results of the paper, including the characterization of peak rules and the relationship between strategy-proofness and the tops-only property. As happens with other preference domains, in our model every strategy-proof social choice rule ignores all information about preferences except individuals' most preferred alternatives. The proof of this property constitutes a major step in establishing our characterization, and we devote considerable space to developing the formal argument that proves this result. Section 4 analyzes the robustness of our results to preference reports outside the single-crossing domain. Final remarks appear in Section 5. For expositional convenience, some of the proofs are in the appendices.

2. The model

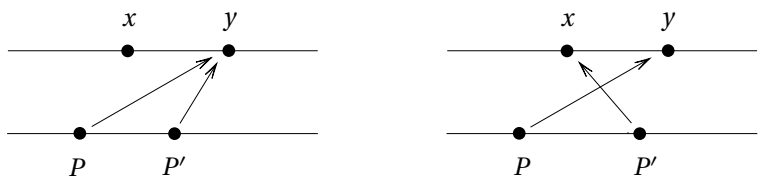
2.1 *Individuals, alternatives and preferences*

Let $N=\{1,\dots,n\}$ be a finite set of individuals. Except where otherwise noted, $n\geq 2$. Let $X=\{x,y,z,\dots\}$ be a finite set of alternatives, with $|X|>2$.[5]

Let $\mathscr{P}$ be the set of all complete, transitive and antisymmetric binary relations on X. A preference ordering over the elements of X is represented by an element P of $\mathscr{P}$, with the usual interpretation that for any pair $x,y\in X$, "$x\,P\,y$" denotes a strict preference for x over y. Sometimes we write $P=(x\,y\,z\,\dots)$ to indicate that $x\,P\,y$, $y\,P\,z$, etc. For any $P\in\mathscr{P}$, and any $Y\subseteq X$, let $\tau|_Y(P)=\arg\max_Y(P)$ be the *top* (*peak*) of P on Y. Notice that the top set $\tau|_Y(P)$ is a singleton because preferences are antisymmetric. For simplicity, we denote $\tau(P)=\tau|_X(P)$.

[4]A set of preference relations with the single-crossing property is maximal if there does not exist another set of preferences that contains the former set and that satisfies single-crossing (see Definition 2, Section 2.1).

[5]For every set A, $|A|$ stands for the cardinality of the set, and $\bar{A}$ for the complement of A.



(a) Condition $SC1$ (b) Violation of $SC1$ and $SC2$

FIGURE 1. Illustration of Definition 1.

DEFINITION 1. A set of preferences $\mathscr{S} \subset \mathscr{P}$ exhibits the *single-crossing property* on X if there is a linear order $>$ of X and a linear order $\succ$ of $\mathscr{S}$ such that $\forall x, y \in X$ and $\forall P, P' \in \mathscr{S}$,

$$[y > x, P' \succ P, \ \& \ y \, P \, x] \Rightarrow y \, P' \, x, \qquad SC1$$

and

$$[y > x, P' \succ P, \ \& \ x \, P' \, y] \Rightarrow x \, P \, y.^{6} \qquad SC2$$

To help the reader gain more insight about this property, Figure 1(a) offers a graphical illustration of condition $SC1$. Figure 1(b) exhibits a case where neither $SC1$ nor $SC2$ is satisfied. In both graphs, arrows denote "preference direction," so that, for example, an arrow from P to y in the presence of x stands for "$y \, P \, x$".

In words, a set of preference relations $\mathscr{S}$ on the set of alternatives X exhibits the single-crossing property (or, for conciseness, $\mathscr{S}$ is single-crossing) if there is a linear order $>$ of X and a linear order $\succ$ of $\mathscr{S}$ such that whenever any preference relation $P \in \mathscr{S}$ ranks any alternative y above (respectively, below) any other alternative x and $y > x$, then so does every other preference relation $P' \in \mathscr{S}$ for which $P' \succ P$ (respectively, $P' \prec P$).

The single-crossing property is closely related to other preference restrictions, such as *hierarchical adherence* (Roberts 1977), *intermediateness* (Grandmont 1978), *order-restriction* (Rothstein 1990, 1991), and *unidimensional alignment* (List 2001).[7] In all of these preference domains the salient feature is the existence of a linear order of the preference relations with the property that, for each pair of alternatives x and y, the relation x preferred to y (or the reverse) partitions the line over which the preferences are ordered into two disjoint intervals. If indifference between alternatives is permitted, then three such intervals arise.

When individuals differ only in their preferences, these domain restrictions can also be defined with respect to an ordering of the agents, instead of the preference relations (see, for example, Rothstein 1990, 1991 and Gans and Smart 1996). That is, the existence of a linear order over the preference relations with the property described above implies

[6]For any $x, y \in X$, we write (1) $x = y$ if and only if $\neg[x > y]$ and $\neg[y > x]$ and (2) $x \geq y$ if and only if either $x = y$ or $x > y$. For any two distinct preferences $P, P' \in \mathscr{S}$, we say that $P \prec P'$ if and only if $\neg[P \succ P']$.

[7]See also Barberà and Moreno (2007), who have recently proposed a weaker condition, called *top-monotonicity*, that encompasses single-crossing, order-restriction, and single-peakedness.

that "we can order individuals in such a way that for any pair of alternatives x and y, the first $j(x,y) \geq 0$ individuals in the ordering strictly prefer x to y (respectively, y to x), the final $k(x,y) \geq 0$ individuals in the ordering strictly prefer y to x (respectively, x to y), and the middle group of individuals, if any, are indifferent between the two" (Austen-Smith and Banks 1999, p. 107).[8]

Scenarios where such a strict ordering of individuals exists are quite common in political economy. "For example, in redistributive politics policy makers are concerned with reallocating resources from rich to poor people, subject to the constraint (typically) that such redistributions do not reverse the rank-order of individuals' wealth. So, while there does not exist an obvious ordering of the alternative distributions of wealth, there does exist a natural ordering of individuals and their preferences in terms of individual wealth" (Austen-Smith and Banks 1999, p. 107).

From a technical perspective, the importance of single-crossing in political economy and public economics is due to the fact that, like single-peakedness, this domain restriction is sufficient to guarantee the existence of a Condorcet winner, especially in cases where the single-peaked property does not hold.[9] However, apart from this, it is worth noting that the conditions are independent, in the sense that neither property is logically implied by the other. Examples 1 and 2 illustrate this point.

Example 1. Consider the set of preference relations $\{P^1, P^2, P^3\}$ in Table 1(a). Recall that, for example, $P^1 = (x\, y\, z)$ stands for $x\, P^1\, y\, P^1\, z$. Simple inspection shows that this set has the single-crossing property on $X = \{x, y, z\}$ with respect to $z > y > x$ and $P^3 \succ P^2 \succ P^1$. However, for every ordering of the alternatives, $\{P^1, P^2, P^3\}$ violates the single-peaked property, because every alternative is ranked bottom in one preference relation. ◊

Example 2. Consider the set of preferences displayed in Table 1(b). This set has the single-peaked property on $X = \{x, y, z, w\}$ with respect to $z > y > x > w$. However, $\{P^1, P^2, P^3\}$ violates Definition 1, because for every ordering of the binary relations and for every ordering of the alternatives, there exist a pair of preference relations in $\{P^1, P^2, P^3\}$ and a pair of alternatives in X such that $SC1$ and $SC2$ are both contradicted. (For example, if $z > y > x > w$, then $P^1 \succ P^3$ contradicts $SC1$ and $SC2$ for the pair $\{x, y\}$, while $P^3 \succ P^1$ does so for $\{z, w\}$.) ◊

Since the main purpose of this paper is to characterize the family of strategy-proof social choice rules on single-crossing domains, in what follows we restrict the analysis to the *largest* or *maximal* sets of preference relations with the single-crossing property. These sets contain the largest number of possible deviations. Therefore, they are the appropriate framework in which to study incentive compatibility.

[8] As the notation indicates, the "cut-off" agents $j(\cdot)$ and $k(\cdot)$ can depend on the pair of alternatives under consideration. In contrast, the order over the preference relations is the same for every pair.

[9] A preference relation $P \in \mathscr{P}$ is *single-peaked* on X if there is a linear order $>$ of X and an alternative $\tau(P) \in X$ such that $\forall x, y \in X$, (i) $\tau(P) > y > x \Rightarrow \tau(P)\, P\, y\, P\, x$, and (ii) $x > y > \tau(P) \Rightarrow \tau(P)\, P\, y\, P\, x$. A set of preference relations $\mathscr{D} \subset \mathscr{P}$ exhibits the *single-peaked property* on X if there is a linear order $>$ of X such that every $P \in \mathscr{D}$ is single-peaked on X with respect to $>$.

$P^1 = (x\,y\,z)$	$P^1 = (x\,y\,z\,w)$
$P^2 = (x\,z\,y)$	$P^2 = (z\,y\,x\,w)$
$P^3 = (z\,y\,x)$	$P^3 = (y\,x\,w\,z)$
a. Single-crossing	b. Single-peakedness

Table 1. Single-crossing and single-peakedness

Definition 2. A set of preferences $\mathscr{S}$ with the single-crossing property on X is *maximal* if there does not exist $\mathscr{S}' \subset \mathscr{P}$ such that $\mathscr{S} \subset \mathscr{S}'$ and $\mathscr{S}'$ exhibits the single-crossing property on X.

Example 3. To illustrate Definition 2, consider again Example 1. Notice that the set of preference relations $\{P^1, P^2, P^3\}$ is not the largest set that satisfies Definition 1 on $X = \{x, y, z\}$, because there exists a preference $P^4 = (z\,x\,y)$ such that $\{P^1, P^2, P^3, P^4\}$ is single-crossing with respect to $z > y > x$ and $P^3 \succ P^4 \succ P^2 \succ P^1$. It is easy to verify that $\{P^1, P^2, P^3, P^4\}$ is indeed maximal. However, it is not unique. If we consider the preference relations $P^5 = (y\,x\,z)$ and $P^6 = (y\,z\,x)$, then the set $\{P^1, P^5, P^6, P^3\}$ is also single-crossing with respect to $z > y > x$ for $P^3 \succ P^6 \succ P^5 \succ P^1$. Moreover, the union of $\{P^1, P^5, P^6, P^3\}$ and $\{P^1, P^2, P^3, P^4\}$ covers all preferences on X. ◊

At this point, it may be useful to compare the size of the set of all single-peaked preferences and the size of the maximal sets with the single-crossing property, for a given ordering of X.[10] The former is well-known to be $2^{|X|-1}$ (see, for instance, Monjardet 2009, p. 144 and the references therein). For single-crossing, the largest size is $|X|\cdot(|X|-1)/2+1$, which is much smaller. To see why, draw a line for each pair of distinct alternatives in X. Observe that, under single-crossing, for each pair $a, b \in X$, the relation a preferred to b (or the reverse) partitions the line associated with $\{a, b\}$ into two disjoint intervals: one interval where the preferences for which a is preferred to b are ordered and a second interval where the relations with the opposite ranking of a and b are ordered (see Figure 2 for the case where $X = \{x, y, z\}$). There are $|X|\cdot(|X|-1)/2$ such partitions. The projection of these partitions into a line forms at most $|X|\cdot(|X|-1)/2+1$ different subintervals. In each subinterval, the preference relation is fully determined. Hence, the given number $|X|\cdot(|X|-1)/2+1$ is an upper bound for the cardinality of the maximal sets of preferences with the single-crossing property.[11]

For the rest of the analysis fix a maximal set $\mathscr{S} \subset \mathscr{P}$ of preference relations with the single-crossing property on X with respect to $>$ and $\succ$. Suppose each individual $i \in N$ is endowed with a preference $P_i \in \mathscr{S}$. Let P_i be agent i's private information. Assume everybody knows the set $\mathscr{S}$; everybody knows that every agent has a preference on X in $\mathscr{S}$; and so on. The n-fold Cartesian product $\mathscr{S}^n$ of $\mathscr{S}$ is the set of all single-crossing preference profiles. As usual, for any profile $\mathbf{P} = (P_1, \ldots, P_n) \in \mathscr{S}^n$, let $\mathbf{P}_{-i} =$

[10] As we note in Example 3, there may be several maximal sets of single-crossing preferences for a given ordering of X. In contrast, the set of all single-peaked preferences is unique once alternatives are ordered.

[11] I am grateful to Hervé Moulin for making this observation in personal correspondence.

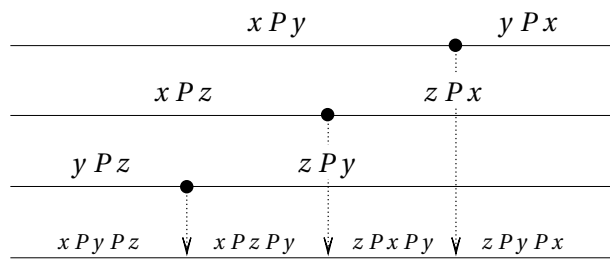


FIGURE 2. Maximal sets of single-crossing preferences.

$(P_1,\dots,P_{i-1},P_{i+1},\dots,P_n)$; for each $\hat{P}_i \in \mathscr{S}$, denote $(\hat{P}_i,\mathbf{P}_{-i}) = (P_1,\dots,P_{i-1},\hat{P}_i,P_{i+1},\dots,P_n)$; and, for every set $S \subseteq N$, let $\mathbf{P}_S = (P_i)_{i\in S}$.

2.2 *Aggregation process*

The problem for the society described in Section 2.1 is to make a social choice from the set of alternatives X. Each individual is entitled to report a preference relation on X from the set of admissible preferences $\mathscr{S}$, which is assumed to be commonly known. These reports are intended to provide information about the profile of true preferences, although agents' sincerity cannot be ensured.

A *social choice rule* is a single-valued mapping $f : \mathscr{S}^n \to X$ that associates to each preference profile $\mathbf{P} \in \mathscr{S}^n$ a unique outcome $f(\mathbf{P}) \in X$. Denote the range of f by $r_f = \{x \in X : \exists\, \mathbf{P} \in \mathscr{S}^n \text{such that} f(\mathbf{P}) = x\}$. Given a social choice rule $f : \mathscr{S}^n \to X$, a nonempty set $S \subset N$, and a profile $\mathbf{P}_{\bar{S}} \in \mathscr{S}^{|\bar{S}|}$, let

$$O_S^f(\mathbf{P}_{\bar{S}}) = \{x \in X : \exists\, \mathbf{P}_S \in \mathscr{S}^{|S|} \text{ such that } f(\mathbf{P}_S,\mathbf{P}_{\bar{S}}) = x\}$$

be the *option set* of S, given that the remaining individuals in $\bar{S} = N \setminus S$ have reported $\mathbf{P}_{\bar{S}}$.

We are interested in social choice rules that satisfy the following properties on $\mathscr{S}^n$. The main one is that no individual ever has an incentive to misrepresent his preferences.

DEFINITION 3. A social choice rule $f : \mathscr{S}^n \to X$ is *strategy-proof* if $\forall\, i \in N$ and $\forall\, (P_i,\mathbf{P}_{-i}) \in \mathscr{S}^n$, there is no $\hat{P}_i \in \mathscr{S}$ such that $f(\hat{P}_i,\mathbf{P}_{-i})\, P_i\, f(P_i,\mathbf{P}_{-i})$.

In words, a social choice rule f on $\mathscr{S}^n$ is strategy-proof (SP) if for any individual $i \in N$, any possible preference $P_i \in \mathscr{S}$ for i, and *any* collection of preferences $\mathbf{P}_{-i} \in \mathscr{S}^{n-1}$ that the other individuals could report, individual i is not better off, according to P_i, by reporting a preference $\hat{P}_i \in \mathscr{S}$ different from P_i. If a social choice rule f is not strategy-proof, then there must exist one agent, say $i \in N$, who can be strictly better off in at least one case, say at $(P_i,\mathbf{P}_{-i}) \in \mathscr{S}^n$, by announcing a preference $\hat{P}_i \in \mathscr{S}$ different from his true ordering P_i. In that case, we say f is *manipulable* by $i \in N$ at $(P_i,\mathbf{P}_{-i}) \in \mathscr{S}^n$ via $\hat{P}_i \in \mathscr{S}$.

To study the possibility of group deviations, it is also possible to define the concept of *group strategy-proofness* (GSP).

Definition 4. A social choice rule $f : \mathscr{S}^n \to X$ is *group strategy-proof* if $\forall S \subseteq N$ and $\forall (\mathbf{P}_S, \mathbf{P}_{\bar{S}}) \in \mathscr{S}^n$, there is no $\hat{\mathbf{P}}_S \in \mathscr{S}^{|S|}$ such that $\forall i \in S$, $f(\hat{\mathbf{P}}_S, \mathbf{P}_{\bar{S}}) P_i f(\mathbf{P}_S, \mathbf{P}_{\bar{S}})$.

Another property that we may seek in a social choice rule is *unanimity* (UN). This property ensures that, if all agents have the same most preferred alternative, then that alternative is socially selected.

Definition 5. A social choice rule $f : \mathscr{S}^n \to X$ is *unanimous* if $\forall x \in X$ and $\forall \mathbf{P} \in \mathscr{S}^n$ such that $\tau(P_i) = x \ \forall i \in N$, $f(\mathbf{P}) = x$.

A profile $\mathbf{P} \in \mathscr{S}^n$ is a permutation of another profile $\hat{\mathbf{P}} \in \mathscr{S}^n$ if there is a one-to-one function $\sigma : N \to N$ such that for every $i \in N$, P_i is identical to $\hat{P}_{\sigma(i)}$. That is, $\mathbf{P}$ is a permutation of $\hat{\mathbf{P}}$ if the lists of preferences under $\mathbf{P}$ and $\hat{\mathbf{P}}$ are identical up to a renaming of the agents.

Definition 6. A social choice rule $f : \mathscr{S}^n \to X$ is *anonymous* if $\forall \mathbf{P}, \hat{\mathbf{P}} \in \mathscr{S}^n$, $f(\mathbf{P}) = f(\hat{\mathbf{P}})$ if $\mathbf{P}$ is a permutation of $\hat{\mathbf{P}}$.

In words, a social choice rule is anonymous (AN) if the names of the individuals holding particular preferences are immaterial in deriving social choices. Notice that since $\mathscr{S}^n$ is a Cartesian product domain, if a profile $\mathbf{P}$ belongs to $\mathscr{S}^n$, then all of its permutations are also in $\mathscr{S}^n$. Thus, the anonymity axiom is non-vacuous in our framework.

The last property of a social choice rule that we consider is the *tops-only* property (TO). We say that f is tops-only if for any admissible preference profile, the social choice is exclusively determined by the individuals' most preferred alternatives on the range of the social choice rule.

Definition 7. A social choice rule $f : \mathscr{S}^n \to X$ is *tops-only* if $\forall \mathbf{P}, \hat{\mathbf{P}} \in \mathscr{S}^n$ such that $\tau|_{r_f}(P_i) = \tau|_{r_f}(\hat{P}_i) \ \forall i \in N$, $f(\mathbf{P}) = f(\hat{\mathbf{P}})$.

The tops-only property severely constrains the scope for manipulation. No agent can expect to be able to affect the social outcome without modifying the peak on the range of his reported preference. Perhaps not surprisingly, we show later in Proposition 2 that this condition is closely related to strategy-proofness, in the sense that every strategy-proof social choice rule on a maximal single-crossing domain is tops-only.[12] The next remark, which follows immediately from Definition 7, is useful in the proof of Proposition 2.

Remark 1. A social choice rule $f : \mathscr{S}^n \to X$ is tops-only if and only if $\forall i \in N$, $\forall (P_i, \mathbf{P}_{-i}) \in \mathscr{S}^n$, and $\forall \hat{P}_i \in \mathscr{S}$ such that $\tau|_{r_f}(\hat{P}_i) = \tau|_{r_f}(P_i)$, $f(P_i, \mathbf{P}_{-i}) = f(\hat{P}_i, \mathbf{P}_{-i})$.

Now we define a class of social choice rules that plays a crucial role in Section 3. To do so, we introduce the following notation. For any odd positive integer k, we say that $m^k : X^k \to X$ is the *k-median* function on X^k if for each $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$,

[12] A similar result holds when preferences are single-peaked, since every strategy-proof social choice rule whose range is an interval satisfies tops-only. See, for instance, Weymark (2008) and Ching (1997).

$|\{x_i : m^k(\mathbf{x}) \geq x_i\}| \geq (k+1)/2$ and $|\{x_j : x_j \geq m^k(\mathbf{x})\}| \geq (k+1)/2$. Since k is odd, $m^k(\mathbf{x})$ is always well defined.

Definition 8. A social choice rule $f : \mathscr{S}^n \to X$ is an *extended median rule* if there are $n+1$ fixed ballots $\alpha_1, \ldots, \alpha_{n+1} \in X$ such that for every preference profile $\mathbf{P} \in \mathscr{S}^n$, $f(\mathbf{P}) = m^{2n+1}(\tau(P_1), \ldots, \tau(P_n), \alpha_1, \ldots, \alpha_{n+1})$.

We denote by f^e a social choice rule that satisfies Definition 8, and by *EMR* the family of all such rules. A particular case of interest within this family is the well-known *median choice rule*, denoted f^m. This rule is obtained from *EMR* by assigning $(n+1)/2$ fixed ballots at $\underline{X} = \min X$ and the rest at $\overline{X} = \max X$, if n is odd, and $n/2$ at $\underline{X}$ and $n/2+1$ at $\overline{X}$ if n is even. Note that, when n is even, f^m breaks the ties in favor of the largest median peak. Alternatively, f^m could break the ties in favor of the smallest median peak, by placing $n/2+1$ fixed ballots at $\underline{X}$ and the remaining $n/2$ at $\overline{X}$.

Proceeding in a similar way, we can derive other rules from *EMR* by restricting each α_i to a particular value of X. For example, if $\alpha_i = \alpha \in X$ for all $i = 1, \ldots, n+1$, then f^e is completely insensitive to the preferences reported by the individuals. We might want to exclude such undesirable rules and, in particular, require Pareto efficiency. A social choice rule $f : \mathscr{S}^n \to X$ is *Pareto efficient* if $\forall \mathbf{P} \in \mathscr{S}^n$ there is no $y \in X$ such that $\forall i \in N$, $y \, P_i \, f(\mathbf{P})$. To eliminate the possibility of inefficiency, we set $\alpha_n = \underline{X}$ and $\alpha_{n+1} = \overline{X}$. By doing so, we derive a social choice rule f^* with the property that for all $\mathbf{P} \in \mathscr{S}^n$, $f^*(\mathbf{P}) = m^{2n-1}(\tau(P_1), \ldots, \tau(P_n), \alpha_1, \ldots, \alpha_{n-1})$. This rule is called an *efficient extended median rule*, and it is characterized by $n-1$ fixed ballots located on X. The set of all such rules is denoted by *EMR**.

Finally, we can also restrict each α_i to take its value at the peak of a preference. We call these rules the *peak rules* and denote the family of all such rules by *PR*.

Definition 9. A social choice rule $f : \mathscr{S}^n \to X$ is a *peak rule* if there are $n-1$ fixed ballots $\alpha_1, \ldots, \alpha_{n-1} \in \{\tau(P) \in X : P \in \mathscr{S}\}$ such that for every preference profile $\mathbf{P} \in \mathscr{S}^n$, $f(\mathbf{P}) = m^{2n-1}(\tau(P_1), \ldots, \tau(P_n), \alpha_1, \ldots, \alpha_{n-1})$.

In the next section, we prove that the set of peak rules is the only family of social choice rules that satisfies unanimity, anonymity, and strategy-proofness on a maximal single-crossing domain. We show also that this characterization is tight, in the sense that relaxing any of the previous conditions enlarges the family of social choice rules.

3. Characterization

We start by proving that every peak rule is group strategy-proof.

Proposition 1. *Every peak rule $f \in PR$ is group strategy-proof on $\mathscr{S}^n$.*

Proof. Fix $f \in PR$. Suppose, by contradiction, there exists a coalition $S \subseteq N$, a profile $(\mathbf{P}_S, \mathbf{P}_{\bar{S}}) \in \mathscr{S}^n$, and a joint deviation $\hat{\mathbf{P}}_S \in \mathscr{S}^{|S|}$ for S such that $f(\hat{\mathbf{P}}_S, \mathbf{P}_{\bar{S}}) \, P_i \, f(\mathbf{P}_S, \mathbf{P}_{\bar{S}})$ for all $i \in S$. To simplify, denote $x = f(\mathbf{P}_S, \mathbf{P}_{\bar{S}})$ and $y = f(\hat{\mathbf{P}}_S, \mathbf{P}_{\bar{S}})$, and let $y > x$.

By definition, $f \in PR$ implies that for all $i = 1, \dots, n-1$, $\alpha_i \in \{\tau(P) \in X : P \in \mathscr{S}\}$. Hence, x and y must coincide with the tops of two preferences. Denote these preferences by P^x and P^y, respectively. We show next that, for all $i \in S$, $\tau(P_i) > x$. Suppose not. That is, assume $x \geq \tau(P_i)$ for some agent $i \in S$. If $\tau(P_i) = x$, then $x \, P_i \, y$, which contradicts our initial hypothesis. Instead, suppose that $x > \tau(P_i)$. Then, $P_i \succ P^x$. Otherwise, $y > x$, $P^x \succ P_i$ and $y \, P_i \, x$ would imply that $y \, P^x \, x$, contradicting the assumption that $x = \tau(P^x)$. But then, by $SC1$, $x \, P^x \, \tau(P_i)$ implies $x \, P_i \, \tau(P_i)$, a contradiction. Therefore, $\forall i \in S$, $\tau(P_i) > x$.

By definition,

$$x = m^{2n-1}(\{\tau(P_i)\}_{i \in S}, \{\tau(P_j)\}_{j \in \bar{S}}, \alpha_1, \dots, \alpha_{n-1})$$

and

$$y = m^{2n-1}(\{\tau(\hat{P}_i)\}_{i \in S}, \{\tau(P_j)\}_{j \in \bar{S}}, \alpha_1, \dots, \alpha_{n-1}).$$

Hence there must exist $i \in S$ for which $x > \tau(\hat{P}_i)$. Otherwise, if $\tau(\hat{P}_i) \geq x \ \forall i \in S$, we would have $y = x$ because $\tau(P_i) > x \ \forall i \in S$. Thus, if we rename $(\{\tau(\hat{P}_i)\}_{i \in S}, \{\tau(P_j)\}_{j \in \bar{S}}, \alpha_1, \dots, \alpha_{n-1})$ as $(a_1, \dots, a_{2n-1})$, it follows that $|\{j \in \{1, \dots, (2n-1)\} : x \geq a_j\}| \geq n$. But then $x \geq m^{2n-1}(a_1, \dots, a_{2n-1})$. That is, $f(\mathbf{P}_S, \mathbf{P}_{\bar{S}}) \geq f(\hat{\mathbf{P}}_S, \mathbf{P}_{\bar{S}})$, contradicting the assumption that $y > x$. Therefore, f is GSP on $\mathscr{S}^n$. □

As we said in the Introduction, in an influential work Moulin (1980) proved that every extended median rule is strategy-proof on the domain of single-peaked preferences. In contrast, Proposition 1 shows only that every peak rule is group strategy-proof (and, consequently, strategy-proof) on any single-crossing domain. Other extended median rules, which allow a fixed ballot to be located over an alternative that is not the peak of a preference, are not guaranteed to be strategy-proof with single-crossing preferences. The following example illustrates this claim.

Example 4. Consider a society with three agents, $N = \{1, 2, 3\}$, and three alternatives, $X = \{x, y, z\}$. Let $\mathscr{S} = \{(x \, y \, z), (x \, z \, y), (z \, x \, y), (z \, y \, x)\}$. As we said in Example 3, these preferences constitute a maximal set with the single-crossing property. Fix a social choice rule $f \in EMR^*$ and assume that $\alpha_1 = y$ and $\alpha_2 = z$. Note that α_1 does not coincide with the most preferred alternative for any preference. Moreover, if $P_1 = (x \, y \, z)$, $P_2 = (x \, z \, y)$, and $P_3 = (z \, y \, x)$, then $f(\mathbf{P}) = m^5(x, x, z, \alpha_1, \alpha_2) = y$. Thus, individual 2, who prefers that the group's choice be either x or z instead of alternative y, can manipulate f by declaring the insincere preference $\hat{P}_2 = (z \, y \, x)$. This causes the outcome to become $f(\hat{P}_2, \mathbf{P}_{-2}) = m^5(x, z, z, \alpha_1, \alpha_2) = z$. Therefore, agent 2's deviation is profitable and individual manipulation cannot be excluded. ◊

This example shows that strategy-proofness is not assured for *every* efficient extended median rule because, with the exception of the subclass of peak rules, all of the other extended median rules do not guarantee that the chosen alternative is always the most preferred alternative for a preference. However, as the proof of Proposition 1 illustrates, this information is used in a fundamental way to rule out orderings that may create incentives for manipulation.

The reason lies in the fact that the single-crossing property is a restriction on the distribution of preferences across individuals, but does not exclude a priori any preference relation. Thus, to get rid of the undesirable orderings, i.e., those which provide incentives to misrepresent the true preferences, the argument cannot rely on the shape of each individual preference, as happens in the case of single-peakedness. Instead, the proof of Proposition 1 shows that the argument exploits (i) the fact that the social choice the most preferred alternative for some preference, (ii) the structure and location of that ordering, and (iii) the correlation between preferences in a set with the single-crossing property. Remarkably, no information about the shape of the preference relations is necessary to guarantee strategy-proofness.

Of course, the conjecture that only peak rules are not manipulable on a maximal single-crossing domain stands in sharp contrast with the main result with the single-peakedness restriction, where every extended median rule (not just peak rules) is strategy-proof. In the next theorem, we formalize this conjecture by showing that the family of peak rules can be characterized by strategy-proofness, anonymity, and unanimity.

Theorem 1. *Let $\mathscr{S}$ be a maximal set of single-crossing preferences. A social choice rule $f : \mathscr{S}^n \to X$ is unanimous, anonymous, and strategy-proof if and only if f is a peak rule.*

The proof of this theorem, which is given in Appendix C, relies on three main results, each important in its own right. The first result, summarized in Proposition 2 and proved in Appendix B, shows that on a maximal set of single-crossing preferences the tops-only property is implied by strategy-proofness. This result is a major step in the proof of Theorem 1, and is consistent with other results in the literature on strategy-proofness. In short, it captures the intuitive idea that social choice rules that use too much information about individuals' preferences are easy to manipulate.

Proposition 2. *Let $\mathscr{S}$ be a maximal set of single-crossing preferences. A social choice rule $f : \mathscr{S}^n \to X$ is strategy-proof only if f is tops-only.*

The proof of Theorem 1 involves two further results in addition to Proposition 2, namely Lemmas 1 and 2. The first lemma points out that if a social choice rule is unanimous and strategy-proof (and therefore tops-only), then no individual must be able to profit by reporting extreme preference relations, unless such extreme preferences constitute the individual's true ordering. This "median property" at the individual level must simultaneously hold for every agent.

To present this result more formally, in the sequel we use $\underline{P}$ (respectively, $\overline{P}$) to denote the most leftist (respectively, rightist) preference relation on X according to the linear order of X, so that for all $x, y \in X$, $x \,\underline{P}\, y$ (respectively, $y \,\overline{P}\, x$) if and only if $y > x$.[13] It is easy to check that these rankings always belong to $\mathscr{S}$. Moreover, $\tau(\underline{P}) = \underline{X}$ and $\tau(\overline{P}) = \overline{X}$.

[13]Obviously, for any $i \in N$, we denote by $\underline{P}_i$ (respectively, $\overline{P}_i$) agent i's preference $\underline{P}$ (respectively, $\overline{P}$).

Lemma 1. *Let $\mathscr{S}$ be a maximal set of single-crossing preferences. A social choice rule $f : \mathscr{S}^n \to X$ is unanimous and strategy-proof only if for all $i \in N$ and all $\mathbf{P} \in \mathscr{S}^n$,*

$$f(P_i, \mathbf{P}_{-i}) = m^3(\tau(P_i), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})).$$

The proof of Lemma 1, carried out below for didactic reasons, is useful to illustrate how the structure of the single-crossing domain is used to justify the existence of certain preferences that are needed in several of the remaining proofs.

Proof of Lemma 1. Let f be UN and SP on $\mathscr{S}^n$. By Proposition 2, f is TO on $\mathscr{S}^n$. Fix a profile $\mathbf{P} \in \mathscr{S}^n$ and an individual $i \in N$. If $f(\underline{P}_i, \mathbf{P}_{-i}) > f(\overline{P}_i, \mathbf{P}_{-i})$, then $f(\underline{P}_i, \mathbf{P}_{-i}) \overline{P}_i f(\overline{P}_i, \mathbf{P}_{-i})$. Thus, agent i can manipulate f at $(\overline{P}_i, \mathbf{P}_{-i})$ via $\underline{P}_i$, a contradiction. Hence, $f(\overline{P}_i, \mathbf{P}_{-i}) \geq f(\underline{P}_i, \mathbf{P}_{-i})$.

There are two cases to consider: (1) $f(\underline{P}_i, \mathbf{P}_{-i}) \geq \tau(P_i)$ and (2) $f(\overline{P}_i, \mathbf{P}_{-i}) > \tau(P_i) > f(\underline{P}_i, \mathbf{P}_{-i})$. The remaining case where $\tau(P_i) \geq f(\overline{P}_i, \mathbf{P}_{-i})$ is similar to (1).

Case 1: $f(\underline{P}_i, \mathbf{P}_{-i}) \geq \tau(P_i)$. Then, $m^3(\tau(P_i), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})) = f(\underline{P}_i, \mathbf{P}_{-i})$. Assume, by way of contradiction, that $f(\mathbf{P}) \neq f(\underline{P}_i, \mathbf{P}_{-i})$. First, suppose that $f(\underline{P}_i, \mathbf{P}_{-i}) > f(\mathbf{P})$. Then, by the definition of $\underline{P}$, we would have $f(\mathbf{P}) \underline{P}_i f(\underline{P}_i, \mathbf{P}_{-i})$, contradicting SP. Second, suppose that $f(\mathbf{P}) > f(\underline{P}_i, \mathbf{P}_{-i})$, which implies that $f(P_i, \mathbf{P}_{-i}) > \tau(P_i)$. By SP, $f(P_i, \mathbf{P}_{-i}) P_i f(\underline{P}_i, \mathbf{P}_{-i})$. Hence, $\tau(P_i) \neq f(\underline{P}_i, \mathbf{P}_{-i})$. Furthermore, $f(\underline{P}_i, \mathbf{P}_{-i}) \neq \tau(\underline{P}_i)$ because $f(\underline{P}_i, \mathbf{P}_{-i}) > \tau(P_i) \geq \tau(\underline{P}_i) = \underline{X}$. In fact, as can be inferred from Figure 3(a), $f(\underline{P}_i, \mathbf{P}_{-i}) \neq \tau(P_j)$ for all $j \neq i$. Otherwise, if $f(\underline{P}_i, \mathbf{P}_{-i}) = \tau(P_j)$ for some $j \in N \setminus \{i\}$, then $P_j \succ P_i$ because $f(\underline{P}_i, \mathbf{P}_{-i}) > \tau(P_i)$. However, by $SC2$, $P_j \succ P_i$, $f(P_i, \mathbf{P}_{-i}) > f(\underline{P}_i, \mathbf{P}_{-i})$, and $f(\underline{P}_i, \mathbf{P}_{-i}) P_j f(P_i, \mathbf{P}_{-i})$ would imply $f(\underline{P}_i, \mathbf{P}_{-i}) P_i f(P_i, \mathbf{P}_{-i})$, a contradiction.

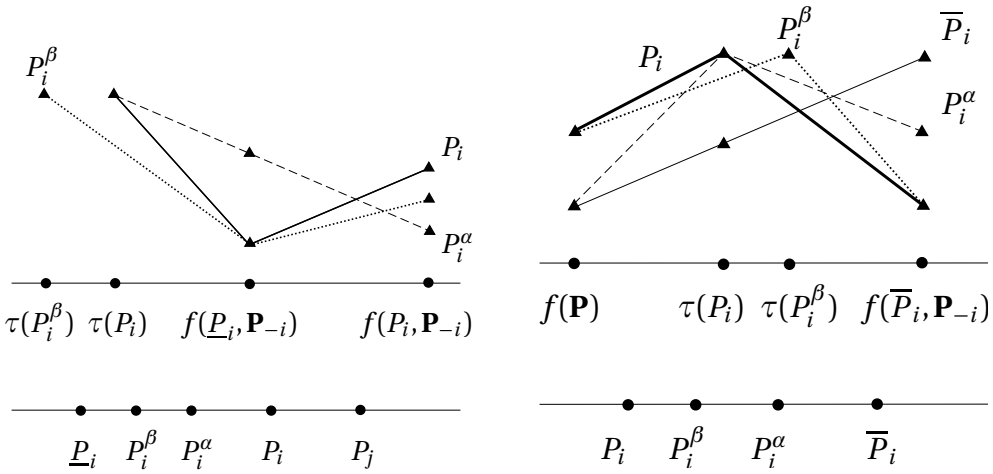
Step 1. Consider any preference $P_i^\alpha \in \mathscr{P}$ such that (i) $\tau(P_i^\alpha) = \tau(P_i)$ and (ii) $f(\underline{P}_i, \mathbf{P}_{-i}) P_i^\alpha f(P_i, \mathbf{P}_{-i})$ (see Figure 3(a)). If $P_i^\alpha \in \mathscr{S}$, then by TO, $f(P_i^\alpha, \mathbf{P}_{-i}) = f(P_i, \mathbf{P}_{-i})$.[14] Using the definition of P_i^α, we have $f(\underline{P}_i, \mathbf{P}_{-i}) P_i^\alpha f(P_i^\alpha, \mathbf{P}_{-i})$, which contradicts SP.

Step 2. If $P_i^\alpha \notin \mathscr{S}$, then there must exist a preference $P_i^\beta \in \mathscr{S}$ that prevents P_i^α from being part of $\mathscr{S}$. Specifically, as we show in Appendix A, there has to be a $P_i^\beta \in \mathscr{S}$ such that (i) $\tau(P_i) > \tau(P_i^\beta)$ and (ii) $f(P_i, \mathbf{P}_{-i}) P_i^\beta f(\underline{P}_i, \mathbf{P}_{-i})$ (see Figure 3(a)).[15] Clearly, P_i must be above P_i^β because the ideal point $\tau(P_i)$ is greater than $\tau(P_i^\beta)$.

Step 3. If $f(\underline{P}_i, \mathbf{P}_{-i}) > f(P_i^\beta, \mathbf{P}_{-i})$, then individual i can manipulate f at $(\underline{P}_i, \mathbf{P}_{-i})$ via P_i^β because, by the definition of $\underline{P}_i$, a smaller alternative is always preferred. Equally, if $f(P_i^\beta, \mathbf{P}_{-i}) = f(\underline{P}_i, \mathbf{P}_{-i})$, then i can manipulate f at $(P_i^\beta, \mathbf{P}_{-i})$ via P_i because, by the definition of P_i^β, $f(P_i, \mathbf{P}_{-i}) P_i^\beta f(\underline{P}_i, \mathbf{P}_{-i})$. Hence $f(P_i^\beta, \mathbf{P}_{-i}) > f(\underline{P}_i, \mathbf{P}_{-i})$. Furthermore, $f(P_i, \mathbf{P}_{-i}) \geq f(P_i^\beta, \mathbf{P}_{-i})$. On the contrary, assume $f(P_i^\beta, \mathbf{P}_{-i}) > f(P_i, \mathbf{P}_{-i})$. Recall that $P_i \succ P_i^\beta$ and, by SP, $f(P_i^\beta, \mathbf{P}_{-i}) P_i^\beta f(P_i, \mathbf{P}_{-i})$. Thus, by $SC1$, $f(P_i^\beta, \mathbf{P}_{-i}) P_i f(P_i, \mathbf{P}_{-i})$, a contradiction. To summarize, $f(P_i, \mathbf{P}_{-i}) \geq f(P_i^\beta, \mathbf{P}_{-i}) > f(\underline{P}_i, \mathbf{P}_{-i})$.

[14] Note that unanimity implies that, for all $i \in N$, and all $P_i \in \mathscr{S}$, $\tau(P_i) \in r_f$. Hence, $\tau(P_i) = \tau|_{r_f}(P_i)$.

[15] The reader should bear in mind that in the figures in which there are P_i^α–P_i^β pairs (with a common integer possibly added to α and β), it is not possible for each preference in the pair to be in the domain. Thus, showing them both ordered on the same line in the diagrams is only for the purpose of the explanation.



(a) Illustration of Case 1. (b) Illustration of Case 2.

FIGURE 3. Illustrations of Cases 1 and 2.

Step 4. Repeating Step 1, suppose that there is a preference $P_i^{\alpha+1} \in \mathscr{S}$ between $\underline{P}_i$ and P_i^β such that (i) $\tau(P_i^{\alpha+1}) = \tau(P_i^\beta)$ and (ii) $f(\underline{P}_i, \mathbf{P}_{-i}) P_i^{\alpha+1} f(P_i^\beta, \mathbf{P}_{-i})$. By TO, $f(P_i^{\alpha+1}, \mathbf{P}_{-i}) = f(P_i^\beta, \mathbf{P}_{-i})$. Hence, $f(\underline{P}_i, \mathbf{P}_{-i}) P_i^{\alpha+1} f(P_i^{\alpha+1}, \mathbf{P}_{-i})$, contradicting the assumption that f is SP.

If instead $P_i^{\alpha+1} \notin \mathscr{S}$, then repeating the argument of Step 2, there must exist $P_i^{\beta+1} \in \mathscr{S}$ with $P_i^{\beta+1} \prec P_i^\beta$ such that (i) $\tau(P_i^\beta) > \tau(P_i^{\beta+1})$ and (ii) $f(P_i^\beta, \mathbf{P}_{-i}) P_i^{\beta+1} f(\underline{P}_i, \mathbf{P}_{-i})$. Using the argument of Step 3, $f(P_i^\beta, \mathbf{P}_{-i}) \geq f(P_i^{\beta+1}, \mathbf{P}_{-i}) > f(\underline{P}_i, \mathbf{P}_{-i})$.

Notice that since X is a finite set and in each step the peak of the "blocking ordering" $P_i^{\beta+k}$ becomes smaller and smaller, the sequence $\tau(P_i^\beta), \tau(P_i^{\beta+1}), \ldots$ approaches $\tau(\underline{P}_i)$. Thus, if we continue applying Steps 1 to 3 repeatedly, then either (i) we eventually get the desired contradiction or (ii) after a finite number of repetitions, say ℓ, we obtain a preference $P_i^{\beta+\ell} \in \mathscr{S}$ between $\underline{P}_i$ and $P_i^{\beta+\ell-1}$ such that $\tau(P_i^{\beta+\ell}) = \tau(\underline{P}_i)$ and $f(P_i^{\beta+\ell-1}, \mathbf{P}_{-i}) P_i^{\beta+\ell} f(\underline{P}_i, \mathbf{P}_{-i})$. By TO, $f(P_i^{\beta+\ell}, \mathbf{P}_{-i}) = f(\underline{P}_i, \mathbf{P}_{-i})$. Therefore, i can manipulate f at $(P_i^{\beta+\ell}, \mathbf{P}_{-i})$ via $P_i^{\beta+\ell-1}$.

Case 2: $f(\overline{P}_i, \mathbf{P}_{-i}) > \tau(P_i) > f(\underline{P}_i, \mathbf{P}_{-i})$. Then $m^3(\tau(P_i), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})) = \tau(P_i)$. Assume, by way of contradiction, that $f(\mathbf{P}) \neq \tau(P_i)$. Without loss of generality, suppose that $\tau(P_i) > f(\mathbf{P})$, so that $f(\overline{P}_i, \mathbf{P}_{-i}) > f(\mathbf{P})$. By SP, $f(P_i, \mathbf{P}_{-i}) P_i f(\overline{P}_i, \mathbf{P}_{-i})$.

Step 1. Consider any preference $P_i^\alpha \in \mathscr{P}$ such that (i) $\tau(P_i^\alpha) = \tau(P_i)$ and (ii) $f(\overline{P}_i, \mathbf{P}_{-i}) P_i^\alpha f(P_i, \mathbf{P}_{-i})$ (see Figure 3(b)). If $P_i^\alpha \in \mathscr{S}$, we are done: by TO, $f(P_i^\alpha, \mathbf{P}_{-i}) = f(P_i, \mathbf{P}_{-i})$. Thus, by the definition of P_i^α, $f(\overline{P}_i, \mathbf{P}_{-i}) P_i^\alpha f(P_i^\alpha, \mathbf{P}_{-i})$, which contradicts SP.

Step 2. On the contrary, if $P_i^\alpha \notin \mathscr{S}$, then using a reasoning analogous to the reasoning of Appendix A, P_i^α must be blocked by a preference $P^* \in \mathscr{S}$ with the property that (i) $\tau(P^*) > \tau(P_i)$ and (ii) $f(P_i, \mathbf{P}_{-i}) P^* f(\overline{P}_i, \mathbf{P}_{-i})$. That is, there must be a preference P^* in $\mathscr{S}$ that is more leftist than P_i^α with respect to the pair $\{f(P_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})\}$ and more

rightist regarding $\{\tau(P^*), \tau(P_i)\}$. Let $P_i^\beta = \min_\succ\{P' \in \mathscr{S} : \tau(P') > \tau(P_i)\}$ (see Figure 3(b)). It is easy to verify that $f(P_i, \mathbf{P}_{-i}) P_i^\beta f(\overline{P}_i, \mathbf{P}_{-i})$, because either P_i^β coincides with P^* or $P^* \succ P_i^\beta$.

Step 3. If $f(P_i^\beta, \mathbf{P}_{-i}) > f(\overline{P}_i, \mathbf{P}_{-i})$, then agent i can manipulate f at $(\overline{P}_i, \mathbf{P}_{-i})$ via P_i^β because, by the definition of $\overline{P}_i$, a greater alternative is always preferred. Equally, if $f(P_i^\beta, \mathbf{P}_{-i}) = f(\overline{P}_i, \mathbf{P}_{-i})$, then i can manipulate f at $(P_i^\beta, \mathbf{P}_{-i})$ via P_i because, by the definition of P_i^β, $f(P_i, \mathbf{P}_{-i}) P_i^\beta f(\overline{P}_i, \mathbf{P}_{-i})$. Hence, $f(\overline{P}_i, \mathbf{P}_{-i}) > f(P_i^\beta, \mathbf{P}_{-i})$. Furthermore, $f(P_i^\beta, \mathbf{P}_{-i}) \geq f(P_i, \mathbf{P}_{-i})$. To see why, assume $f(P_i, \mathbf{P}_{-i}) > f(P_i^\beta, \mathbf{P}_{-i})$. Recall that $P_i \prec P_i^\beta$ and, by SP, $f(P_i^\beta, \mathbf{P}_{-i}) P_i^\beta f(P_i, \mathbf{P}_{-i})$. Thus, by $SC2$, $f(P_i^\beta, \mathbf{P}_{-i}) P_i f(P_i, \mathbf{P}_{-i})$, a contradiction. To summarize, $f(\overline{P}_i, \mathbf{P}_{-i}) > f(P_i^\beta, \mathbf{P}_{-i}) \geq f(P_i, \mathbf{P}_{-i})$.

Step 4. Suppose $f(P_i^\beta, \mathbf{P}_{-i}) = \tau(P_i^\beta)$. Then $f(P_i^\beta, \mathbf{P}_{-i}) > f(P_i, \mathbf{P}_{-i})$ because $\tau(P_i^\beta) > \tau(P_i) > f(P_i, \mathbf{P}_{-i})$. Since $\mathscr{S}$ is maximal, there is $P_i^{\alpha'} \in \mathscr{S}$ with the property that $\tau(P_i^{\alpha'}) = \tau(P_i)$ and $f(P_i^\beta, \mathbf{P}_{-i}) P_i^{\alpha'} f(P_i, \mathbf{P}_{-i})$. In effect, to prevent $P_i^{\alpha'}$ from belonging to $\mathscr{S}$, there should be $P^* \in \mathscr{S}$ such that $\tau(P^*) > \tau(P_i)$ and $f(P_i, \mathbf{P}_{-i}) P^* f(P_i^\beta, \mathbf{P}_{-i})$, which is not possible because, by SP, $f(P_i^\beta, \mathbf{P}_{-i}) P_i^\beta f(P_i, \mathbf{P}_{-i})$ (remember that $P_i^\beta = \min_\succ\{P' \in \mathscr{S} : \tau(P') > \tau(P_i)\}$). By TO, $f(P_i^{\alpha'} \mathbf{P}_{-i}) = f(P_i, \mathbf{P}_{-i})$; hence, agent i can manipulate f at $(P_i, \mathbf{P}_{-i})$ via P_i^β, a contradiction. Therefore, $f(P_i^\beta, \mathbf{P}_{-i}) \neq \tau(P_i^\beta)$.

Step 5. Repeating Step 1, suppose that there is a preference $P_i^{\alpha+1} \in \mathscr{S}$ between $\overline{P}_i$ and P_i^β such that (i) $\tau(P_i^{\alpha+1}) = \tau(P_i^\beta)$ and (ii) $f(\overline{P}_i, \mathbf{P}_{-i}) P_i^{\alpha+1} f(P_i^\beta, \mathbf{P}_{-i})$. By TO, $f(P_i^{\alpha+1}, \mathbf{P}_{-i}) = f(P_i^\beta, \mathbf{P}_{-i})$, and we are done; i.e., agent i can manipulate f at $(P_i^{\alpha+1}, \mathbf{P}_{-i})$ via $\overline{P}_i$.

On the contrary, if $P_i^{\alpha+1} \notin \mathscr{S}$, then by the argument of Step 2, there must exist a preference $P_i^{\beta+1} \in \mathscr{S}$ with $P_i^{\beta+1} \succ P_i^\beta$ such that (i) $\tau(P_i^{\beta+1}) > \tau(P_i^\beta)$ and (ii) $f(P_i^\beta, \mathbf{P}_{-i}) P_i^{\beta+1} f(\overline{P}_i, \mathbf{P}_{-i})$. Using the reasoning of Step 3, $f(\overline{P}_i, \mathbf{P}_{-i}) > f(P_i^{\beta+1}, \mathbf{P}_{-i}) \geq f(P_i^\beta, \mathbf{P}_{-i})$.

If we go back to Step 1 and continue applying Steps 1 to 4 repeatedly, then in the end either (i) we get the desired contradiction or (ii) after a finite number of repetitions, say ℓ, we find $P_i^{\beta+\ell} \in \mathscr{S}$ between $\overline{P}_i$ and $P_i^{\beta+\ell-1}$ such that $\tau(P_i^{\beta+\ell}) = \tau(\overline{P}_i)$ and $f(P_i^{\beta+\ell-1}, \mathbf{P}_{-i}) P_i^{\beta+\ell} f(\overline{P}_i, \mathbf{P}_{-i})$. By TO, $f(P_i^{\beta+\ell}, \mathbf{P}_{-i}) = f(\overline{P}_i, \mathbf{P}_{-i})$. Therefore, i can manipulate f at $(P_i^{\beta+\ell}, \mathbf{P}_{-i})$ via $P_i^{\beta+\ell-1}$, contradicting the assumption that f is SP. □

Finally, the proof of Theorem 1 also uses Lemma 2, according to which a strategy-proof and unanimous social choice rule must satisfy a property called *top-monotonicity* (TM). Roughly speaking, this property ensures that collective choices do not respond perversely to changes in individuals' ideal points.

Definition 10. A social choice rule $f : \mathscr{S}^n \to X$ is *top-monotonic* if $\forall i \in N$, $\forall (P_i, \mathbf{P}_{-i}) \in \mathscr{S}^n$, and $\forall \hat{P}_i \in \mathscr{S}$ such that $\tau(\hat{P}_i) \geq \tau(P_i)$, $f(\hat{P}_i, \mathbf{P}_{-i}) \geq f(P_i, \mathbf{P}_{-i})$.

Lemma 2. *Let $\mathscr{S}$ be a maximal set of single-crossing preferences. If a social choice rule $f : \mathscr{S}^n \to X$ is unanimous and strategy-proof, then f is top-monotonic.*

Proof. Let f be UN and SP on $\mathscr{S}^n$. Consider any individual $i \in N$, any profile $(P_i, \mathbf{P}_{-i}) \in$

$\mathscr{S}^n$, and any admissible deviation $P_i' \in \mathscr{S}$ such that $\tau(P_i') \geq \tau(P_i)$. We want to show that $f(P_i', \mathbf{P}_{-i}) \geq f(P_i, \mathbf{P}_{-i})$. Three cases are possible.

Case 1: If $\tau(P_i) \geq f(\overline{P}_i, \mathbf{P}_{-i})$, then

$$m^3(\tau(P_i), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})) = m^3(\tau(P_i'), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i}))$$

because SP implies that $f(\overline{P}_i, \mathbf{P}_{-i}) \geq f(\underline{P}_i, \mathbf{P}_{-i})$ and, by hypothesis, $\tau(P_i') \geq \tau(P_i)$. Therefore, by Lemma 1, $f(P_i', \mathbf{P}_{-i}) = f(P_i, \mathbf{P}_{-i})$.

Case 2: If $f(\overline{P}_i, \mathbf{P}_{-i}) > \tau(P_i) > f(\underline{P}_i, \mathbf{P}_{-i})$, then $m^3(\tau(P_i), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})) = \tau(P_i)$, and, given that $\tau(P_i') \geq \tau(P_i)$, $m^3(\tau(P_i'), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})) \geq \tau(P_i)$. Therefore, by Lemma 1, $f(P_i', \mathbf{P}_{-i}) \geq f(P_i, \mathbf{P}_{-i})$.

Case 3: Finally, if $f(\underline{P}_i, \mathbf{P}_{-i}) \geq \tau(P_i)$, then by Lemma 1, $f(P_i', \mathbf{P}_{-i}) \geq f(P_i, \mathbf{P}_{-i})$ because $m^3(\tau(P_i'), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})) \geq m^3(\tau(P_i), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})) = f(\underline{P}_i, \mathbf{P}_{-i})$. □

Under the hypotheses of Theorem 1, the social choice always coincides with the peak of a preference. Thus, a corollary that can be immediately established is that, on a maximal set of single-crossing preferences, every UN, AN, and SP social choice rule satisfies Pareto efficiency.

Corollary 1. *Let $\mathscr{S}$ be a maximal set of single-crossing preferences. If a social choice rule $f : \mathscr{S}^n \to X$ is unanimous, anonymous, and strategy-proof, then f is Pareto efficient.*

Proof. Fix a UN, AN and SP social choice rule $f : \mathscr{S}^n \to X$. By Theorem 1, f is a peak rule. Without loss of generality, suppose there exist $x, y \in X$, $y > x$, and a profile $\mathbf{P} \in \mathscr{S}^n$ such that $f(\mathbf{P}) = x$ and $y \; P_i \; x$ for all $i \in N$. Then, for all $i = 1, \ldots n$, $\tau(P_i) \neq x$. By Theorem 1, there exists $P^x \in \mathscr{S}$ such that $\tau(P^x) = x$. For all $i \in N$, $P_i \succ P^x$. Otherwise, $SC1$ would imply $y \; P^x \; x$. Therefore, $\min_{>}\{\tau(P_1), \ldots, \tau(P_n)\} > x$ and, by Definition 9, $f(\mathbf{P}) > x$, a contradiction. Thus, f is Pareto efficient. □

In addition to showing Corollary 1, under the hypotheses of Theorem 1 it is also possible to show that the set of admissible preferences has the single-peaked property over the range of the social choice rule. More formally, for any set $Y \subset X$ and any preference $P \in \mathscr{S}$, let $P|_Y$ be the restriction of the relation P to the elements of Y. Denote by $\mathscr{S}|_Y$ the set containing the restriction of each preference $P \in \mathscr{S}$ to Y. We refer to $\mathscr{S}|_Y$ as the restriction of $\mathscr{S}$ to Y.

Lemma 3. *Let $\mathscr{S}$ be a maximal set of single-crossing preferences. If a social choice rule $f : \mathscr{S}^n \to X$ is unanimous, anonymous, and strategy-proof, then the restriction of $\mathscr{S}$ to the range of f has the single-peaked property.*

Proof. The proof is based on Saporiti and Tohmé (2006). Fix a maximal set $\mathscr{S} \subset \mathscr{P}$ with the single-crossing property with respect to $>$ and $\succ$. Take a UN, AN, and SP social choice rule $f : \mathscr{S}^n \to X$. Assume, by contradiction, there exists a preference $P|_{r_f} \in \mathscr{S}|_{r_f}$ that is not single-peaked on r_f with respect to the linear order $>$ of X. Then there must

be a triple $x, y, z \in r_f$ such that $x > y > z$ and $x\,P\,y$ and $z\,P\,y$. By Theorem 1, $y = \tau|_{r_f}(P')$ for some $P' \in \mathscr{S}$. If $P' \succ P$, then by $SC1$, $x > y$ and $x\,P\,y$ imply $x\,P'\,y$, contradicting the assumption that $y = \tau|_{r_f}(P')$. Hence, $P \succ P'$. However, since $y > z$ and $z\,P\,y$, $SC2$ implies $z\,P'\,y$, a contradiction. Therefore, the set $\mathscr{S}|_{r_f}$ has the single-peaked property on r_f. □

A social choice rule $f : \mathscr{S}^n \to X$ has a *regular domain* if for every $\alpha \in r_f$ there is a preference $P^\alpha \in \mathscr{S}$ such that $\tau|_{r_f}(P^\alpha) = \alpha$ (Le Breton and Weymark 1999). As we state in Corollary 2, another immediate consequence of Theorem 1 is that every UN, AN, and SP social choice rule $f : \mathscr{S}^n \to X$ has a regular domain. The result follows immediately from Theorem 1 and Definition 9.

Corollary 2. *Let $\mathscr{S}$ be a maximal set of single-crossing preferences. If a social choice rule $f : \mathscr{S}^n \to X$ is unanimous, anonymous, and strategy-proof, then $\mathscr{S}^n$ is a regular domain.*

Finally, we close this section by discussing the independence of the axioms used in Theorem 1, as well as the role of the maximal domain condition. First, consider the consequence of relaxing strategy-proofness. As we have explained, any efficient extended median rule that it is not a peak rule may be subject to individual manipulation on a single-crossing domain. However, all such rules are anonymous and unanimous. Thus, the family that satisfies these two axioms on $\mathscr{S}^n$ is larger than the set of peak rules.

Second, consider the consequence of relaxing unanimity. Define a social choice rule f in such a way that, for each $\mathbf{P} \in \mathscr{S}^n$, $f(\mathbf{P}) = a \in X$. It is clear that f is anonymous and strategy-proof. However, f violates unanimity since $r_f = \{a\}$. Hence, $f \notin PR$.

Third, relax anonymity by fixing an agent $j \in N$ and defining a social choice rule f in such a way that, for all $\mathbf{P} \in \mathscr{S}^n$, $f(\mathbf{P}) = \tau(P_j)$. It is immediate to see that f is unanimous and strategy-proof. However, it violates anonymity because f is dictatorial.

Lastly, to illustrate why the maximal domain condition is needed to derive the main results of this paper, let $N = \{1, 2\}$, $X = \{x, y, z\}$ with $z > y > x$, and $\mathscr{S} = \{\underline{P}, \overline{P}\}$, where $\underline{P} = (x\,y\,z)$ and $\overline{P} = (z\,y\,x)$. As is clear from Example 3, the set of preferences $\mathscr{S}$ is not a maximal set with the single-crossing property. Define f by setting $f(\underline{P}, \underline{P}) = x$, $f(\overline{P}, \overline{P}) = z$, and $f(\underline{P}, \overline{P}) = f(\overline{P}, \underline{P}) = y$. This function satisfies unanimity, anonymity, and strategy-proofness. However, f is not a peak rule because y is chosen at $(\underline{P}, \overline{P})$ and at $(\overline{P}, \underline{P})$, but $y \neq \tau(P)$ for all $P \in \mathscr{S} = \{\underline{P}, \overline{P}\}$.

4. Robustness

So far, we have assumed that every individual $i \in N$ is endowed with a preference P_i drawn from the restricted domain $\mathscr{S}$ and is entitled to report a preference relation (not necessarily the true one) from the same admissible set. That is, we have restricted *both* the true preferences of all individuals and their strategies, i.e., the orderings they are permitted to announce, to the *same* maximal set of single-crossing preferences. The main result obtained from this assumption is that a social choice rule is anonymous, unanimous, and strategy-proof if and only if it is a peak rule.

There are two main concerns regarding this result. First, there is a question of how easy it is to describe the set of admissible preferences. We said in the Introduction that the set of individual preferences with the single-crossing property can be derived from standard assumptions in economics. However, to be able to describe this set, the mechanism designer would probably need to possess some information about individuals' preferences, though not about any particular individual's ordering. Given that the goal of this paper is to study social choice problems where individual preferences are privately observed, the information required by the planner to specify the set of admissible reports weakens the contribution of Theorem 1.

Nevertheless, as Campbell and Kelly (2003, p. 567) say, "there is a sense in which results based on a domain of single-peaked preferences have the same drawback: Although single-peaked domains can be defined as product sets, single-peakedness is characterized by means of a particular linear ordering, and an individual would have to know the linear ordering to which the reported preference is admissible, before being convinced that his own reported preference is admissible."

Furthermore, while in some cases this ordering is *natural* and, therefore, the assumption that it is commonly known (included by the planner) is not too demanding, in others it is not necessarily obvious. Suppose, for example, that alternatives are political candidates. Then, the way in which individuals agree to locate these candidates on a one-dimensional political scale is not immediate. Moreover, that ordering not only determines which preferences can be declared, but it also provides information about other individuals' preferences. For instance, if $X = \{x, y, z\}$ and $(P_i)_{i \in N}$ is single-peaked with respect to $x > y > z$, then the order of the alternatives reveals that nobody holds a preference that ranks y bottom (such as the relations $P = (x\, z\, y)$ and $P' = (z\, x\, y)$).

Apart from the difficulty of specifying the set of possible reports, a second concern is that, even if the mechanism designer has the information to do so, it is still unclear how to deal with declarations that are not in the admissible set. Can we tell an individual that, despite the fact that preferences are not directly observed, on the basis of our beliefs about "how they should be," he cannot submit a certain preference relation because we consider it somehow "unreasonable" and, therefore, it has been removed from the set of possible declarations?

Once again, this affects not only the analysis with single-crossing preferences, but also with other domain restrictions. Consider, for instance, the case where preferences satisfy the single-peaked property over the real line. For the planner, it would not be difficult to describe the set of admissible preferences because alternatives are ordered according to the usual order of the real numbers. However, suppose that individual i reports a preference that is not single-peaked on that order. What can we do in such a situation? Can we say to individual i that he is not entitled to have such a preference relation? In a free society, every individual can order the alternatives in the way he wishes, independently of how sensible we think these orderings are. Thus, assuming that preference relations that do not satisfy the domain restriction will not be permitted seems neither realistic nor democratic.

To deal with this problem, in this section we analyze the possibility of strengthening the result of Theorem 1 by eliminating the requirement that individual reports be restricted to be in the set $\mathscr{S}$. The analysis is inspired by Blin and Satterthwaite (1976), who have undertaken a similar exercise to assess the robustness of the strategy-proof result of majority rule with Borda completion on the domain of single-peaked preferences when individual reports are allowed to be outside the single-peaked domain.

Our findings are positive: If the true preferences of the society satisfy the single-crossing property, then no individual can ever profitably manipulate a peak rule by reporting a preference that is not his true preference relation, independently of whether the insincere preference belongs to the single-crossing domain or not. Conversely, if we allow deviations outside the single-crossing domain, every anonymous, unanimous, and strategy-proof social choice rule must be a peak rule on the set of preferences with the single-crossing property.

To see this more formally, let us now redefine a social choice rule so that it associates a feasible alternative to every profile of complete, transitive, and antisymmetric preferences; i.e., let $f : \mathscr{P}^n \to X$. Following Blin and Satterthwaite (1976), a social choice rule f is said to be manipulable on $\mathscr{S}^n$ if there exist an individual $i \in N$, a profile $(P_i, \mathbf{P}_{-i}) \in \mathscr{S}^n$, and a deviation $\hat{P}_i \in \mathscr{P}$ such that $f(\hat{P}_i, \mathbf{P}_{-i}) P_i f(P_i, \mathbf{P}_{-i})$. A social choice rule is strategy-proof on $\mathscr{S}^n$ if and only if it is not manipulable on $\mathscr{S}^n$. Notice that, when we defined strategy-proofness in Definition 3, we omitted the qualification "on $\mathscr{S}^n$" because that is also the domain of the social choice rule. Instead, here the social choice rule is defined on a larger domain, actually on the set of all strict preferences; but it is required to satisfy strategy-proofness only on the domain of individual true preferences.

Proceeding in a similar way, we can redefine unanimity and anonymity. A social choice rule $f : \mathscr{P}^n \to X$ is unanimous on $\mathscr{S}^n$ if $\forall x \in X$ and $\forall \mathbf{P} \in \mathscr{S}^n$ such that $\tau(P_i) = x \ \forall i \in N$, $f(\mathbf{P}) = x$. Similarly, $f : \mathscr{P}^n \to X$ is anonymous on $\mathscr{S}^n$ if for all $\mathbf{P}, \hat{\mathbf{P}} \in \mathscr{S}^n$, $f(\mathbf{P}) = f(\hat{\mathbf{P}})$ whenever $\mathbf{P}$ is a permutation of $\hat{\mathbf{P}}$. Finally, we say that a social choice rule $f : \mathscr{P}^n \to X$, defined over the set of all complete, transitive, and antisymmetric preference profiles, is a peak rule on $\mathscr{S}^n$ if there are $n-1$ fixed ballots $\alpha_1, \dots, \alpha_{n-1} \in \{\tau(P) \in X : P \in \mathscr{S}\}$ such that $\forall \mathbf{P} \in \mathscr{S}^n$, $f(\mathbf{P}) = m^{2n-1}(\tau(P_1), \dots, \tau(P_n), \alpha_1, \dots, \alpha_{n-1})$.

Theorem 2. *Every peak rule $f : \mathscr{P}^n \to X$ on $\mathscr{S}^n$ is strategy-proof on $\mathscr{S}^n$. Conversely, if $\mathscr{S}$ is a maximal set of single-crossing preferences, then every social choice rule $f : \mathscr{P}^n \to X$ that is unanimous, anonymous, and strategy-proof on $\mathscr{S}^n$ is a peak rule on $\mathscr{S}^n$.*

Proof. To prove the first part of the theorem, fix any peak rule $f : \mathscr{P}^n \to X$ on $\mathscr{S}^n$. Suppose that f is manipulable by $i \in N$ at a profile $(P_i, \mathbf{P}_{-i}) \in \mathscr{S}^n$ via a preference relation $\hat{P}_i \in \mathscr{P}$ that is not necessarily in $\mathscr{S}$. Without loss of generality, suppose that $f(\hat{P}_i, \mathbf{P}_{-i}) > f(P_i, \mathbf{P}_{-i})$. Since f always chooses the most preferred alternative of a preference, let $f(P_i, \mathbf{P}_{-i})$ coincide with $\tau(P^*)$ for some $P^* \in \mathscr{S}$. By $SC1$, $(P_i, \mathbf{P}_{-i}) \in \mathscr{S}^n$ and $f(\hat{P}_i, \mathbf{P}_{-i}) P_i f(P_i, \mathbf{P}_{-i})$ imply that $P_i \succ P^*$. Therefore, $\tau(P_i) > f(P_i, \mathbf{P}_{-i})$. Moreover, $f(P_i, \mathbf{P}_{-i}) > \tau(\hat{P}_i)$. Otherwise, we would have $f(\hat{P}_i, \mathbf{P}_{-i}) = f(P_i, \mathbf{P}_{-i})$. Hence, by the definition of f, $f(P_i, \mathbf{P}_{-i}) \geq f(\hat{P}_i, \mathbf{P}_{-i})$, which contradicts the initial hypothesis that $f(\hat{P}_i, \mathbf{P}_{-i}) > f(P_i, \mathbf{P}_{-i})$. Therefore, f is SP on $\mathscr{S}^n$.

The proof of the second part of the theorem is immediate. Consider a social choice rule $f : \mathscr{P}^n \to X$ that is UN, AN, and SP on $\mathscr{S}^n$. Define the social choice rule $g : \mathscr{S}^n \to X$ in such a way that, for all $\mathbf{P} \in \mathscr{S}^n$, $g(\mathbf{P}) = f(\mathbf{P})$. Since g is AN, UN, and SP, by Theorem 1, $g \in PR$. Hence, f is a peak rule on $\mathscr{S}^n$. □

Notice that the first part of Theorem 2 is proved applying the same argument used in the proof to Proposition 1. This is because the latter does not exploit the structure of the deviation profile $(\hat{P}_i, \mathbf{P}_{-i})$. On the contrary, what matters is that $(P_i, \mathbf{P}_{-i})$ belongs to $\mathscr{S}^n$. The second part of the theorem holds because every social choice rule that is not manipulable when individuals can report *any* strict preference relation must also not be manipulable when they are allowed to declare only preferences from a strictly smaller subset. However, we have already shown in Theorem 1 that, when reports are restricted to $\mathscr{S}$, every unanimous, anonymous, and strategy-proof social choice rule is a peak rule. Hence, if we dispense with the assumption that declarations are restricted to the set of preferences with the single-crossing property, we must obtain the same family of rules on the restricted domain.

Finally, note that Theorem 2 does not provide a full characterization because we have not determined the form of a unanimous, anonymous, and strategy-proof social choice rule outside the domain of preferences with the single-crossing property. However, it does show that the rules obtained in any such characterization coincide over a maximal single-crossing domain with the rules characterized in Theorem 1. This, together with the fact that every peak rule is strategy-proof on the domain of single-crossing preferences, allows us to conclude that the result stated in Theorem 1 is robust to the kind of perturbations introduced in this section.

5. Final remarks

This paper analyzes strategy-proof collective choice rules when individuals have single-crossing preferences on a finite and ordered set of social alternatives. While the single-crossing property has been shown to be sufficient to ensure the existence of a Condorcet winner, this result has been derived assuming that individuals sincerely declare their preferences. This naturally raises the issue of potential individual and group manipulation, motivating the current research.

The main contributions of this paper are the following. First, the paper shows that, in addition to single-peakedness, single-crossing is another meaningful domain that guarantees the existence of strategy-proof social choice rules. Specifically, it proves that every peak rule is group strategy-proof on any set of preferences with the single-crossing property. Conversely, it shows that every social choice rule that satisfies anonymity, unanimity, and strategy-proofness on a maximal single-crossing domain is a member of this family. These results are robust to deviations outside the single-crossing domain, provided that individuals' true preferences belong to that set.

A natural consequence of the previous characterization is that anonymity, unanimity, and strategy-proofness imply Pareto efficiency. Furthermore, although in our framework individual preferences need not be convex over the set of alternatives,

anonymity, unanimity, and strategy-proofness also imply that preferences must satisfy single-peakedness over the range of the social choice rule.[16] Thus, our results indicate that to rule out incentives to misrepresent individual preferences some convexity and regularity of the domain are necessary.

Another important conclusion of this research is that on a maximal single-crossing domain, strategy-proofness implies the tops-only property. The proof of this claim does not completely follow the proof strategy recently proposed by Weymark (2008) because the non-convexities of single-crossing preferences make it quite difficult to directly prove that the Le Breton and Weymark (1999) regularity condition is satisfied. To circumvent this problem, first we prove the claim in a two-person case, where unanimity over the range can be used without further complications. We then obtain a partial characterization of the social choice rule for the case with only two individuals, which allows the claim to be established for more than two agents by reducing the analysis to a situation where only two tops are different.

Finally, this paper also shows that the Representative Voter Theorem has a well defined strategic foundation, in the sense that the median voter's ideal point can be implemented in dominant strategies by a direct mechanism. However, this conclusion holds on a subdomain of single-crossing preferences, the Cartesian product domain. Therefore, relaxing the assumption that individuals sincerely reveal their preferences is not free.

Appendices

A. Further details for the proof of Lemma 1

The second step of the proof of Lemma 1, Case 1, argues that

$$[P_i^\alpha \notin \mathscr{S}] \Rightarrow [\exists P_i^\beta \in \mathscr{S} \text{ such that (i) } \tau(P_i) > \tau(P_i^\beta) \text{ and (ii) } f(P_i, \mathbf{P}_{-i})\, P_i^\beta\, f(\underline{P}_i, \mathbf{P}_{-i})]. \quad (*)$$

If such a preference P_i^β exists in $\mathscr{S}$, it is clear that $P_i \succ P_i^\beta$ because $\tau(P_i) > \tau(P_i^\beta)$. Thus, in what follows we focus on the existence of P_i^β.

Assume that $P_i^\alpha \notin \mathscr{S}$ and suppose, by way of contradiction, that $(*)$ is not true. Recall that we are working under the assumption that $f(P_i, \mathbf{P}_{-i}) > f(\underline{P}_i, \mathbf{P}_{-i})$. That means $\tau(P_i) \neq \tau(\underline{P}_i)$ (remember that f is TO). Thus there exists a preference $P' \in \mathscr{S}$ such that $\tau(P_i) > \tau(P')$. In particular, this is true for $\underline{P}_i$. Hence, it must be that for all $P' \in \mathscr{S}$ with the property that $\tau(P_i) > \tau(P')$, $f(\underline{P}_i, \mathbf{P}_{-i})\, P'\, f(P_i, \mathbf{P}_{-i})$ (otherwise, we would have a preference P_i^β that satisfies $(*)$). Define the set $Z(P_i) = \{P' \in \mathscr{S} : \tau(P_i) > \tau(P') \text{ and } f(\underline{P}_i, \mathbf{P}_{-i})\, P'\, f(P_i, \mathbf{P}_{-i})\}$. Once again, $Z(P_i) \neq \emptyset$ because $\underline{P}_i \in Z(P_i)$. Take the greatest element of $Z(P_i)$ with respect to $\succ$ and denote it by $\max_\succ Z(P_i) \equiv P^{\max}$.

Recall from Step 1 that P_i^α is *any* preference for which (i) $\tau(P_i^\alpha) = \tau(P_i)$ and (ii) $f(\underline{P}_i, \mathbf{P}_{-i})\, P_i^\alpha\, f(P_i, \mathbf{P}_{-i})$. Moreover, by hypothesis, none of these orderings belong to $\mathscr{S}$. (Otherwise, the argument of Step 1 would provide the desired result.) Therefore, it must be that for all $P'' \in \mathscr{S}$ with the property that $\tau(P'') = \tau(P_i)$, $f(P_i, \mathbf{P}_{-i})\, P''\, f(\underline{P}_i, \mathbf{P}_{-i})$.

[16] Given a set X and a linear order $>$ of X, a preference $P \in \mathscr{P}$ on X is convex with respect to $>$ if for every three distinct alternatives $x, y, z \in X$, $x\, P\, y \Rightarrow y\, P\, z$ whenever y is between x and z.

Define the set $Y(P_i) = \{P'' \in \mathscr{S} : \tau(P'') = \tau(P_i) \text{ and } f(P_i, \mathbf{P}_{-i})\, P''\, f(\underline{P}_i, \mathbf{P}_{-i})\}$. Notice that $Y(P_i) \neq \varnothing$ because $P_i \in Y(P_i)$. Take the smallest element of $Y(P_i)$ with respect to $\succ$ and denote it by $\min_{\succ} Y(P_i) \equiv P^{\min}$. Obviously, $P^{\min} \succ P^{\max}$ because $\tau(P^{\min}) = \tau(P_i) > \tau(P^{\max})$.

For every pair of distinct alternatives $x, y \in X$, with $y > x$, notice that either $P^{\max}$ and $P^{\min}$ coincide in ranking x and y, or $x\, P^{\max}\, y$ and $y\, P^{\min}\, x$. This is because $P^{\max}, P^{\min} \in \mathscr{S}$, $P^{\min} \succ P^{\max}$, and $y > x$.

Consider the following mixture P^{α^*} of $P^{\max}$ and $P^{\min}$. For every pair of distinct alternatives $x, y \in X$, with $y > x$, (i) if $x\, P^{\min}\, y$, then $x\, P^{\alpha^*}\, y$; (ii) if $y\, P^{\max}\, x$, then $y\, P^{\alpha^*}\, x$; and (iii) if $x\, P^{\max}\, y$ and $y\, P^{\min}\, x$, then $x\, P^{\alpha^*}\, y$ if $y \neq \tau(P_i)$, and $y\, P^{\alpha^*}\, x$ otherwise. In words, for every pair of distinct alternatives $x, y \in X$, with $y > x$, P^{α^*} ranks x and y like $P^{\max}$ and $P^{\min}$ whenever these two preferences coincide; it ranks x and y like $P^{\max}$ if $P^{\max}$ and $P^{\min}$ do not coincide and the greatest alternative, i.e. y, is not the peak of $P^{\min}$; and, otherwise, P^{α^*} follows $P^{\min}$. Just to illustrate this mixture, it is easy to see that if $P^{\max} = (a\, b\, c\, d)$, $P^{\min} = (b\, d\, c\, a)$, and $d > c > b > a$, then $P^{\alpha^*} = (b\, a\, c\, d)$.

First, note that P^{α^*} is a linear order; hence, $P^{\alpha^*} \in \mathscr{P}$. Second, $f(\underline{P}_i, \mathbf{P}_{-i})\, P^{\alpha^*}\, f(P_i, \mathbf{P}_{-i})$ because $f(P_i, \mathbf{P}_{-i}) > f(\underline{P}_i, \mathbf{P}_{-i})$, $f(P_i, \mathbf{P}_{-i}) \neq \tau(P_i)$, and $f(\underline{P}_i, \mathbf{P}_{-i})\, P^{\max}\, f(P_i, \mathbf{P}_{-i})$. Third, $\tau(P^{\alpha^*}) = \tau(P^{\min}) = \tau(P_i)$. Finally, since $\mathscr{S}$ is maximal, by construction $P^{\alpha^*} \in \mathscr{S}$, contradicting the initial hypothesis that any preference P_i^{α} for which (i) $\tau(P_i^{\alpha}) = \tau(P_i)$ and (ii) $f(\underline{P}_i, \mathbf{P}_{-i})\, P_i^{\alpha}\, f(P_i, \mathbf{P}_{-i})$ does not belong to $\mathscr{S}$. Therefore, $(*)$ holds.

B. Proof of Proposition 2

In order to prove Proposition 2, the following lemma, which is a special case of Proposition 2 in Le Breton and Weymark (1999), is extremely useful.

Lemma 4. *Suppose $f : \mathscr{S}^n \to X$ is a strategy-proof social choice rule with $n \geq 1$. For any nonempty set $S \subseteq N$, any $x \in r_f$, and every profile $(\mathbf{P}_S, \mathbf{P}_{\bar{S}}) \in \mathscr{S}^n$ such that $\tau|_{O_S^f(\mathbf{P}_{\bar{S}})}(P_i) = x$ for all $i \in S$, $f(\mathbf{P}_S, \mathbf{P}_{\bar{S}}) = x$.*[17]

From Lemma 4, we can derive Corollaries 3 and 4, whose proofs follow immediately by setting $S = \{i\}$ and $S = N$, respectively.

Corollary 3. *If $f : \mathscr{S}^n \to X$ is a strategy-proof social choice rule, then for all $i \in N$, every $x \in r_f$, and all $(P_i, \mathbf{P}_{-i}) \in \mathscr{S}^n$ such that $\tau|_{O_i^f(\mathbf{P}_{-i})}(P_i) = x$, $f(P_i, \mathbf{P}_{-i}) = x$.*

Corollary 4. *If $f : \mathscr{S}^n \to X$ is a strategy-proof social choice rule, then for all $x \in r_f$ and all $(P_i, \mathbf{P}_{-i}) \in \mathscr{S}^n$ such that $\tau|_{r_f}(P_i) = x$ for all $i \in N$, $f(P_i, \mathbf{P}_{-i}) = x$.*

In words, Corollary 4 points out that a strategy-proof social choice rule must respect unanimity over the range, in the sense that if everyone has the same most preferred alternative on the range of the social choice rule, then that alternative must be the social choice.

Finally, the following additional result can be derived from Lemma 4.

[17] If $S = N$, it is assumed that $O_N^f(\mathbf{P}_{\varnothing}) = r_f$.

Corollary 5. *If $f : \mathscr{S}^n \to X$ is a strategy-proof social choice rule, then for any nonempty set $S \subset N$, every $x \in r_f$, and all $(\mathbf{P}_S, \mathbf{P}_{\bar{S}}) \in \mathscr{S}^n$ such that $\tau|_{r_f}(P_i) = x$ for all $i \in S$, $x \in O^f_{\bar{S}}(\mathbf{P}_S)$.*

Corollary 5 is immediate from Corollary 4 (just take a preference relation for each individual in the set $\bar{S}$ with the most preferred alternative over the range equal to x). Roughly speaking, it says that if a social choice rule is strategy-proof and all agents in a certain coalition agree on the most preferred alternative over the range of the rule, then that alternative must be available in the option set of the remaining agents.

The next lemma shows that, when there are two individuals in the society, a social choice rule is strategy-proof only if it satisfies the tops-only property. This, in turn, implies that any two preferences in the admissible domain, with the same most preferred alternative over the range of the rule, must have the same top on any option set generated by the preference of the other individual. This is an immediate consequence of Remark 1 and Corollary 3.

Lemma 5. *Let $|N| = 2$ and suppose that $\mathscr{S}$ is a maximal set of single-crossing preferences. A social choice rule $f : \mathscr{S}^2 \to X$ is strategy-proof only if f is tops-only.*

Proof. Assume, by way of contradiction, that there exists a strategy-proof social choice rule $f : \mathscr{S}^2 \to X$ that is not tops-only. By Remark 1, there must exist a profile $(P_1, P_2) \in \mathscr{S}^2$ and a preference $\hat{P}_1 \in \mathscr{S}$ such that $\tau|_{r_f}(\hat{P}_1) = \tau|_{r_f}(P_1)$ and $f(\hat{P}_1, P_2) = y \neq x = f(P_1, P_2)$. By Corollary 3, $\tau|_{O^f_1(P_2)}(P_1) = x$ and $\tau|_{O^f_1(P_2)}(\hat{P}_1) = y$. The rest of the proof consists in showing that this supposition leads to a contradiction with the fact that f is a strategy-proof social choice rule. A similar argument is used in the proof of Lemma 7.

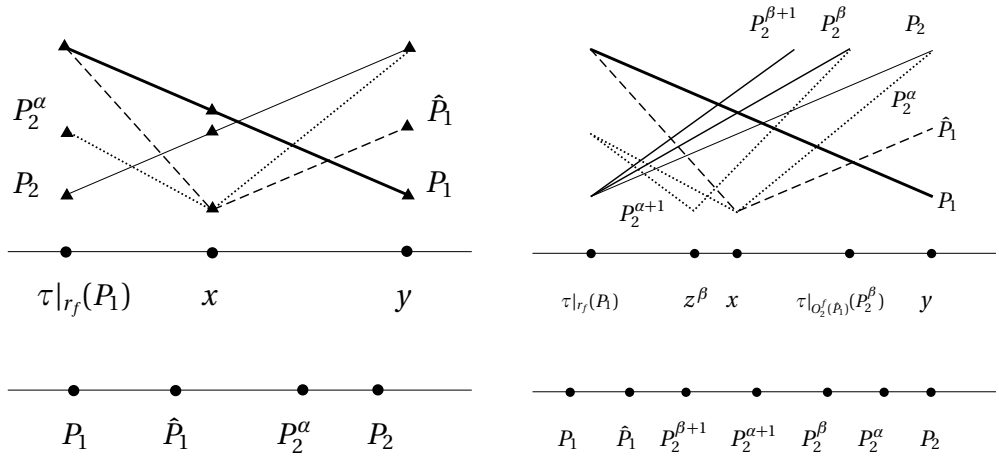
Without loss of generality, assume $\tau|_{O^f_1(P_2)}(\hat{P}_1) = y > x = \tau|_{O^f_1(P_2)}(P_1)$. Hence, $\hat{P}_1 \succ P_1$. Obviously, $x\, P_1\, y$, $y\, \hat{P}_1\, x$, $\tau|_{r_f}(P_1) \neq x$, and $\tau|_{r_f}(\hat{P}_1) \neq y$. Furthermore, note that $x\, P_2\, \tau|_{r_f}(P_1)$. Otherwise, by Corollary 4, agent 2 can manipulate f at (P_1, P_2) via a $\tilde{P}_2$ equal to P_1 (which results in $\tau|_{r_f}(P_1)$ being chosen). Using a similar argument, $y\, P_2\, \tau|_{r_f}(P_1)$.

There are two cases to consider: (1) $y > \tau|_{r_f}(P_1) > x$ and (2) $y > x > \tau|_{r_f}(P_1)$. The remaining situation, i.e., $\tau|_{r_f}(P_1) > y > x$, is similar to the second case.

Case 1: $y > \tau|_{r_f}(P_1) > x$. If $P_2 \succ P_1$, then by $SC2$, $\tau|_{r_f}(P_1) > x$ and $x\, P_2\, \tau|_{r_f}(P_1)$ imply that $x\, P_1\, \tau|_{r_f}(P_1)$, a contradiction. Thus, $P_1 \succ P_2$. But then, by $SC1$, $y > \tau|_{r_f}(P_1)$ and $y\, P_2\, \tau|_{r_f}(P_1)$ imply that $y\, P_1\, \tau|_{r_f}(P_1)$, a contradiction.

Case 2: $y > x > \tau|_{r_f}(P_1)$. First, suppose that $\tau|_{r_f}(P_2) = x$. Then, $P_2 \succ P_1$. Otherwise, by $SC1$, $P_1 \succ P_2$, $x > \tau|_{r_f}(P_1)$, and $x\, P_2\, \tau|_{r_f}(P_1)$ would imply that $x\, P_1\, \tau|_{r_f}(P_1)$. Similarly, $\hat{P}_1 \succ P_2$, since $x\, P_2\, y$ implies $x\, P\, y\, \forall P \prec P_2$, and $y\, \hat{P}_1\, x$ by hypothesis. However, $\tau|_{r_f}(\hat{P}_1)\, \hat{P}_1\, x$ and $\tau|_{r_f}(\hat{P}_1) = \tau|_{r_f}(P_1)$ imply $\tau|_{r_f}(P_1)\, P\, x\, \forall P \prec \hat{P}_1$, contradicting the assumption that $\tau|_{r_f}(P_2) = x$. Hence, for all $j = 1, 2$, $x \neq \tau|_{r_f}(P_j)$.

Second, if $P_1 \succ P_2$, then $\tau|_{r_f}(P_1)\, P_1\, x$ implies $\tau|_{r_f}(P_1)\, P_2\, x$, a contradiction. Thus, $P_2 \succ P_1$ and, therefore, $\tau|_{r_f}(P_2) > \tau|_{r_f}(P_1)$. Similarly, if $\hat{P}_1 \succ P_2$, then $\tau|_{r_f}(P_1)\, \hat{P}_1\, \tau|_{r_f}(P_2)$ implies $\tau|_{r_f}(P_1)\, P_2\, \tau|_{r_f}(P_2)$. So, $P_2 \succ \hat{P}_1$. Moreover, $y\, P_2\, x$, since $x\, P_2\, y$ would imply $x\, \hat{P}_1\, y$. Finally, if $y > \tau|_{r_f}(P_2)$, then $\tau|_{r_f}(P_2)\, P_2\, y$ would imply $\tau|_{r_f}(P_2)\, \hat{P}_1\, y$, contradicting SP because, by Corollary 4, agent 1 would profitably manipulate f at $(\hat{P}_1, P_2)$ via a $\tilde{P}_1$ equal to P_2. Hence, $\tau|_{r_f}(P_2) \geq y$, and we face a situation as in Figure 4(a).



(a) Illustration of Case 2. (b) Illustration of P_2^β, $P_2^{\alpha+1}$ and $P_2^{\beta+1}$.

FIGURE 4. Illustrations of Case 2 and of P_2^β, $P_2^{\alpha+1}$ and $P_2^{\beta+1}$

Note that $y \notin O_2^f(P_1)$. Otherwise, there must be $P_2' \in \mathscr{S}$ such that $f(P_1, P_2') = y$. Because $y \; P_2 \; x$, it would then follow that agent 2 could manipulate f at (P_1, P_2) via P_2'.

Step 1. Consider any preference $P_2^\alpha \in \mathscr{P}$ such that

$$\text{(i) } \tau|_{O_2^f(\hat{P}_1)}(P_2^\alpha) = y = \tau|_{O_2^f(\hat{P}_1)}(P_2) \text{ and (ii) } \tau|_{O_2^f(P_1)}(P_2^\alpha) = \tau|_{r_f}(P_1)$$

(see Figure 4(a)). If $P_2^\alpha \in \mathscr{S}$, by Corollary 3, $f(\hat{P}_1, P_2^\alpha) = y$ and $f(P_1, P_2^\alpha) = \tau|_{r_f}(P_1)$. Hence, agent 1 can manipulate f at $(\hat{P}_1, P_2^\alpha)$ via P_1, contradicting that f is SP, and we are done.

Step 2. If $P_2^\alpha \notin \mathscr{S}$, then applying the reasoning of Appendix A, there must exist a preference $P^* \in \mathscr{S}$ such that

$$\text{(i) } y > \tau|_{O_2^f(\hat{P}_1)}(P^*) \text{ and (ii) } \tau|_{O_2^f(P_1)}(P^*) > \tau|_{r_f}(P_1).$$

That is, if $P_2^\alpha \notin \mathscr{S}$, there has to be P^* in $\mathscr{S}$ such that (i) P^* is more leftist than P_2^α with respect to the top on $O_2^f(\hat{P}_1)$ and (ii) P^* is more rightist than P_2^α with respect to the top on $O_2^f(P_1)$.[18] Let

$$P_2^\beta = \max_{\succ}\{P^* \in \mathscr{S} : y > \tau|_{O_2^f(\hat{P}_1)}(P^*) \text{ and } \tau|_{O_2^f(P_1)}(P^*) > \tau|_{r_f}(P_1)\}.$$

Obviously, $\hat{P}_1 \prec P_2^\beta \prec P_2$ because $\tau|_{O_2^f(\hat{P}_1)}(P_2) = y$ and $\tau|_{O_2^f(P_1)}(\hat{P}_1) = \tau|_{r_f}(P_1)$ (see Figure 4(b)). Denote $\tau|_{O_2^f(P_1)}(P_2^\beta) = z^\beta$.

Step 3. By Corollary 3, $f(\hat{P}_1, P_2^\beta) = \tau|_{O_2^f(\hat{P}_1)}(P_2^\beta)$ and $f(P_1, P_2^\beta) = z^\beta > \tau|_{r_f}(P_1)$. If $\tau|_{O_2^f(\hat{P}_1)}(P_2^\beta) = \tau|_{r_f}(P_1)$, then agent 1 can manipulate f at (P_1, P_2^β) via $\hat{P}_1$. Hence

[18]The other possibility that can prevent P_2^α from being in $\mathscr{S}$ is a preference with a top lower than $\tau|_{r_f}(P_1)$ on $O_2^f(P_1)$ and a top greater than y on $O_2^f(\hat{P}_1)$. However, the existence of such order in $\mathscr{S}$ is ruled out by P_1.

$\tau|_{O_2^f(\hat{P}_1)}(P_2^\beta) \neq \tau|_{r_f}(P_1)$. Furthermore, if $z^\beta > x$, then $P_2 \succ P_2^\beta$ and $z^\beta \, P_2^\beta \, x$ imply $z^\beta \, P_2 \, x$, contradicting the assumption that $x = \tau|_{O_2^f(P_1)}(P_2)$. Therefore, $x \geq z^\beta > \tau|_{r_f}(P_1)$.

Step 4. If $\tau|_{O_2^f(\hat{P}_1)}(P_2^\beta) = z^\beta$, then $z^\beta \, P_2^\beta \, y$ because $y \in O_2^f(\hat{P}_1)$. Moreover, $z^\beta \leq x < y$ and $\hat{P}_1 \prec P_2^\beta$ imply, by $SC2$, that $z^\beta \, \hat{P}_1 \, y$ and $z^\beta \, P_1 \, y$. Since $\mathscr{S}$ is maximal, there is $P_2^{\alpha'} \in \mathscr{S}$ with the property that $\tau|_{O_2^f(\hat{P}_1)}(P_2^{\alpha'}) = y$ and $\tau|_{O_2^f(P_1)}(P_2^{\alpha'}) = z^\beta$. In effect, to prevent $P_2^{\alpha'}$ from belonging to $\mathscr{S}$, there should be $P^* \in \mathscr{S}$ such that $\tau|_{O_2^f(\hat{P}_1)}(P^*) < y = \tau|_{O_2^f(\hat{P}_1)}(P_2^{\alpha'})$ and $\tau|_{O_2^f(P_1)}(P^*) > z^\beta = \tau|_{O_2^f(P_1)}(P_2^{\alpha'})$, which is not possible due to the definition of P_2^β (remember that $P_2^\beta = \max_{\succ}\{P^* \in \mathscr{S} : y > \tau|_{O_2^f(\hat{P}_1)}(P^*) \text{ and } \tau|_{O_2^f(P_1)}(P^*) > \tau|_{r_f}(P_1)\}$). By Corollary 3, $f(P_1, P_2^{\alpha'}) = z^\beta$ and $f(\hat{P}_1, P_2^{\alpha'}) = y$. Hence, agent 1 manipulates f at $(\hat{P}_1, P_2^{\alpha'})$ via P_1, a contradiction. Therefore, $\tau|_{O_2^f(\hat{P}_1)}(P_2^\beta) \neq z^\beta$.

Step 5. Proceeding as in Step 1, consider any preference $P_2^{\alpha+1} \in \mathscr{P}$ such that

$$\text{(i) } \tau|_{O_2^f(\hat{P}_1)}(P_2^{\alpha+1}) = \tau|_{O_2^f(\hat{P}_1)}(P_2^\beta) \text{ and (ii) } \tau|_{O_2^f(P_1)}(P_2^{\alpha+1}) = \tau|_{r_f}(P_1)$$

(see Figure 4(b)). If $P_2^{\alpha+1} \in \mathscr{S}$, by Corollary 3, we have $f(P_1, P_2^{\alpha+1}) = \tau|_{r_f}(P_1)$ and $f(\hat{P}_1, P_2^{\alpha+1}) = \tau|_{O_2^f(\hat{P}_1)}(P_2^\beta) \neq \tau|_{r_f}(P_1)$. Hence, agent 1 can manipulate f at $(\hat{P}_1, P_2^{\alpha+1})$ via P_1, contradicting the assumption that f is SP, and we are done.

If $P_2^{\alpha+1} \notin \mathscr{S}$, then using the reasoning of Step 2, there must exist a preference $P_2^{\beta+1} \in \mathscr{S}$ with $P_2^{\beta+1} \prec P_2^\beta$ such that

$$\text{(i) } \tau|_{O_2^f(\hat{P}_1)}(P_2^\beta) > \tau|_{O_2^f(\hat{P}_1)}(P_2^{\beta+1}) \text{ and (ii) } \tau|_{O_2^f(P_1)}(P_2^{\beta+1}) = z^{\beta+1} > \tau|_{r_f}(P_1)$$

(see Figure 4(b)). By the argument of Step 3, $z^\beta \geq z^{\beta+1} > \tau|_{r_f}(P_1)$.

Going back to Step 1 and performing the analysis repeatedly, in the end either (i) we get the desired contradiction at some point in the process or (ii) after a finite number of repetitions, say ℓ, we eventually arrive at a preference $P_2^{\beta+\ell} \in \mathscr{S}$ between $P_2^{\beta+\ell-1}$ and $\hat{P}_1$ such that

$$\text{(i) } \tau|_{O_2^f(\hat{P}_1)}(P_2^{\beta+\ell}) = \tau|_{r_f}(P_1) \text{ and (ii) } \tau|_{O_2^f(P_1)}(P_2^{\beta+\ell}) = z^{\beta+\ell} > \tau|_{r_f}(P_1).$$

By Corollary 3, $f(\hat{P}_1, P_2^{\beta+\ell}) = \tau|_{r_f}(P_1)$ and $f(P_1, P_2^{\beta+\ell}) = z^{\beta+\ell} \neq \tau|_{r_f}(P_1)$. Hence, agent 1 can manipulate f at $(P_1, P_2^{\beta+\ell})$ via $\hat{P}_1$, a contradiction. Therefore, f is TO on $\mathscr{S}^2$. □

The following corollary follows immediately from Lemma 5 and the argument of Lemma 1.

Corollary 6. *Let $|N| = 2$ and suppose that $\mathscr{S}$ is a maximal set of single-crossing preferences. A social choice rule $f : \mathscr{S}^2 \to X$ is strategy-proof only if for all $i \in N$ and all $\mathbf{P} \in \mathscr{S}^2$, $f(P_i, \mathbf{P}_{-i}) = m^3(\tau|_{r_f}(P_i), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i}))$.*

Now, before generalizing Lemma 5 to the case where $|N| > 2$, we first extend the tops-only property to the option sets generated by a strategy-proof social choice rule. We do

this in two steps. First, we prove in Lemma 6 that the option set of any single individual $i \in N$ satisfies a tops-only property when there is agreement among the individuals in $N \setminus \{i\}$ as to which alternative is best on the range. Then, in Lemma 7, we generalize this result to the option set of any individual when the remaining agents do not necessarily agree on the most preferred alternative over the range.

Lemma 6. *Let $\mathscr{S}$ be a maximal set of single-crossing preferences. If a social choice rule $f : \mathscr{S}^n \to X$ is strategy-proof, then for each individual $i \in N$ and every pair of profiles $\mathbf{P}'_{-i}, \mathbf{P}''_{-i} \in \mathscr{S}^{n-1}$ for which $\tau|_{r_f}(P'_j) = \tau|_{r_f}(P''_k)$ for all $j, k \in N \setminus \{i\}$, $O_i^f(\mathbf{P}'_{-i}) = O_i^f(\mathbf{P}''_{-i})$.*

Proof. Consider any $i \in N$ and any two profiles $\mathbf{P}'_{-i}, \mathbf{P}''_{-i} \in \mathscr{S}^{n-1}$,

$$\mathbf{P}'_{-i} = (P'_1, \dots, P'_{i-1}, P'_{i+1}, \dots, P'_n)$$
$$\mathbf{P}''_{-i} = (P''_1, \dots, P''_{i-1}, P''_{i+1}, \dots, P''_n)$$

such that, for all $j, k \in N \setminus \{i\}$, $\tau|_{r_f}(P'_j) = \tau|_{r_f}(P''_k) = z$ for some $z \in X$. To simplify the notation, assume $P'_j = P'_k$ and $P''_j = P''_k$ for all $j, k \in N \setminus \{i\}$, so that we can write[19]

$$\mathbf{P}'_{-i} = (\underbrace{P', \dots, P'}_{n-1 \text{ times}}) \quad \text{and} \quad \mathbf{P}''_{-i} = (\underbrace{P'', \dots, P''}_{n-1 \text{ times}}).$$

We want to show that $O_i^f(\mathbf{P}'_{-i}) = O_i^f(\mathbf{P}''_{-i})$. To do that, define the sequence of profiles $\mathbf{P}^0_{-i} = (P', \dots, P')$, $\mathbf{P}^1_{-i} = (P'', P', \dots, P')$, $\dots$, $\mathbf{P}^{n-1}_{-i} = (P'', \dots, P'')$. To establish the result, it is enough to prove that, for all $j = 1, \dots, n-1$, $O_i^f(\mathbf{P}^j_{-i}) = O_i^f(\mathbf{P}^{j-1}_{-i})$. Without loss of generality, assume that there exists $x \in X$ such that for some $1 \le j^* \le n-1$, $x \in O_i^f(\mathbf{P}^{j^*-1}_{-i})$ and $x \notin O_i^f(\mathbf{P}^{j^*}_{-i})$. By Corollary 5, $z \in O_i^f(\mathbf{P}^{j^*-1}_{-i}) \cap O_i^f(\mathbf{P}^{j^*}_{-i})$. Therefore $z \neq x$. Moreover, since $x \in O_i^f(\mathbf{P}^{j^*-1}_{-i})$, there exists $\widetilde{P}_i \in \mathscr{S}$ such that $f(\widetilde{P}_i, \mathbf{P}^{j^*-1}_{-i}) = x$.

Notice that the preference profiles $\mathbf{P}^{j^*-1}_{-i}$ and $\mathbf{P}^{j^*}_{-i}$ differ only in one preference relation. Without loss of generality, suppose that it is the preference of agent $\ell \in N \setminus \{i\}$:

$$\mathbf{P}^{j^*-1}_{-i} = (\underbrace{P'', \dots, P''}_{j^*-1}, P'_\ell, \underbrace{P', \dots, P'}_{n-j^*-1})$$
$$\mathbf{P}^{j^*}_{-i} = (\underbrace{P'', \dots, P''}_{j^*-1}, P''_\ell, \underbrace{P', \dots, P'}_{n-j^*-1}).$$

Fix

$$\mathbf{P}^{j^*-1}_{-\{i,\ell\}} = (\underbrace{P'', \dots, P''}_{j^*-1}, \underbrace{P', \dots, P'}_{n-j^*-1}) = \mathbf{P}^{j^*}_{-\{i,\ell\}}.$$

Define the two-person social choice rule $g : \mathscr{S}^2 \to X$ in such a way that, for all $(P_i, P_\ell) \in \mathscr{S}^2$, $g(P_i, P_\ell) = f(P_i, P_\ell, \mathbf{P}^{j^*-1}_{-\{i,\ell\}})$. It is easy to show that g is strategy-proof and $r_g = O^f_{\{i,\ell\}}(\mathbf{P}^{j^*-1}_{-\{i,\ell\}})$. By Corollary 5, $z \in r_g$. Hence, $\tau|_{r_g}(P''_\ell) = \tau|_{r_g}(P'_\ell) = z$ because $r_g \subseteq r_f$ and,

[19] To avoid any misunderstanding, it is worth emphasizing that the fact that the preference relations in the profile $\mathbf{P}'_{-i}$ (respectively, $\mathbf{P}''_{-i}$) are identical does not play any role in the proof. The only thing that matters is that they have a common most preferred alternative over r_f.

by hypothesis, $\tau|_{r_f}(P''_\ell) = \tau|_{r_f}(P'_\ell) = z$. By Lemma 5, g is tops-only. Therefore, $g(\widetilde{P}_i, P''_\ell) = g(\widetilde{P}_i, P'_\ell)$. By the definition of g, $g(\widetilde{P}_i, P'_\ell) = f(\widetilde{P}_i, \mathbf{P}^{j^*-1}_{-i}) = x$. Thus, $g(\widetilde{P}_i, P''_\ell) = x$. That is, $f(\widetilde{P}_i, \mathbf{P}^{j^*}_{-i}) = x$. Therefore, $x \in O^f_i(\mathbf{P}^{j^*}_{-i})$, a contradiction. Hence, $O^f_i(\mathbf{P}'_{-i}) = O^f_i(\mathbf{P}''_{-i})$. □

Lemma 7. *Let $\mathscr{S}$ be a maximal set of single-crossing preferences. If a social choice rule $f : \mathscr{S}^n \to X$ is strategy-proof, then for each $i \in N$ and every pair of profiles $\mathbf{P}'_{-i}, \mathbf{P}''_{-i} \in \mathscr{S}^{n-1}$ such that $\tau|_{r_f}(P'_j) = \tau|_{r_f}(P''_j)$ for all $j \in N \setminus \{i\}$, $O^f_i(\mathbf{P}'_{-i}) = O^f_i(\mathbf{P}''_{-i})$.*

Proof. Consider any individual $i \in N$ and any two profiles $\hat{\mathbf{P}}_{-i}, \check{\mathbf{P}}_{-i} \in \mathscr{S}^{n-1}$ such that for all $j, k \in N \setminus \{i\}$, $\hat{P}_j = \hat{P}_k$, and $\check{P}_j = \check{P}_k$, and for each $j \in N \setminus \{i\}$, $\tau|_{r_f}(\hat{P}_j) = \tau|_{r_f}(\check{P}_j) = z$ for some $z \in X$. By Lemma 6, $O^f_i(\hat{\mathbf{P}}_{-i}) = O^f_i(\check{\mathbf{P}}_{-i})$.

Step 1. Fix any individual $j \neq i$ and any two preferences $P'_j, P''_j \in \mathscr{S}$ such that $\tau|_{r_f}(P'_j) = w = \tau|_{r_f}(P''_j)$ for some $w \in X$. We want to show that $O^f_i(P'_j, \hat{\mathbf{P}}_{-\{i,j\}}) = O^f_i(P''_j, \hat{\mathbf{P}}_{-\{i,j\}})$. Define the two-person social choice rule $g : \mathscr{S}^2 \to X$ in such a way that for all $(P_i, P_j) \in \mathscr{S}^2$, $g(P_i, P_j) = f(P_i, P_j, \hat{\mathbf{P}}_{-\{i,j\}})$. Since f is SP on $\mathscr{S}^n$, g is SP on $\mathscr{S}^2$, with range $r_g = O^f_{\{i,j\}}(\hat{\mathbf{P}}_{-\{i,j\}})$. If $\tau|_{r_g}(P'_j) = \tau|_{r_g}(P''_j)$, then by applying Lemma 6 to g we get $O^g_i(P'_j) = O^g_i(P''_j)$, and we are done; i.e., by the definition of g, $O^f_i(P'_j, \hat{\mathbf{P}}_{-\{i,j\}}) = O^f_i(P''_j, \hat{\mathbf{P}}_{-\{i,j\}})$.

Instead, if $\tau|_{r_g}(P'_j) = a \neq b = \tau|_{r_g}(P''_j)$, then Lemma 6 cannot be used because it relies on the existence of a common peak on the range of the social choice rule. So, we proceed as follows. Without loss of generality, let $b > a$, implying that $P''_j \succ P'_j$. Assume, by way of contradiction, that $O^g_i(P'_j) \neq O^g_i(P''_j)$. That is, suppose that there is $\alpha \in r_g$ such that $\alpha \in O^g_i(P'_j)$ and $\alpha \notin O^g_i(P''_j)$. Hence, there must be a $\widetilde{P}_i \in \mathscr{S}$ such that $g(\widetilde{P}_i, P'_j) = \alpha$. Since $\alpha \notin O^g_i(P''_j)$, let $g(\widetilde{P}_i, P''_j) = \beta \neq \alpha$. By SP, $\alpha\, P'_j\, \beta$ and $\beta\, P''_j\, \alpha$. By single-crossing, $P''_j \succ P'_j$ implies $\beta > \alpha$.

We would like to find two preferences $P^\alpha_i, P^\beta_i \in \mathscr{S}$, not necessarily different, such that (i) $\tau|_{r_f}(P^\alpha_i) = w$, (ii) $g(P^\alpha_i, P'_j) = \alpha$, (iii) $\tau|_{r_f}(P^\beta_i) = w$, and (iv) $g(P^\beta_i, P''_j) = \beta$. We show below that such preferences exist in $\mathscr{S}$. First, note that if $\widetilde{P}_i$ is between P'_j and P''_j, then we already have the desired preferences, because in that case $\tau|_{r_f}(\widetilde{P}_i) = w$. So, without loss of generality, suppose that $\widetilde{P}_i \succ P''_j$, implying that $\tau|_{r_g}(\widetilde{P}_i) = c \geq b$. Clearly, $\beta\, \widetilde{P}_i\, \alpha$ because $\beta\, P''_j\, \alpha$. Therefore, $\tau|_{r_g}(\widetilde{P}_i) = c \neq \alpha$. By Lemma 5, g is TO over r_g. By Corollary 6,

$$g(\widetilde{P}_i, P'_j) = m^3(\tau|_{r_g}(\widetilde{P}_i), g(\underline{P}_i, P'_j), g(\overline{P}_i, P'_j)). \tag{1}$$

Applying Corollary 6 once again to $g(\underline{P}_i, P'_j)$ and to $g(\overline{P}_i, P'_j)$, we get

$$g(\underline{P}_i, P'_j) = m^3(\tau|_{r_g}(P'_j), g(\underline{P}_i, \underline{P}_j), g(\underline{P}_i, \overline{P}_j)) \tag{2}$$

$$g(\overline{P}_i, P'_j) = m^3(\tau|_{r_g}(P'_j), g(\overline{P}_i, \underline{P}_j), g(\overline{P}_i, \overline{P}_j)). \tag{3}$$

By Corollary 4, $g(\underline{P}_i, \underline{P}_j) = \underline{X}_{r_g}$ and $g(\overline{P}_i, \overline{P}_j) = \overline{X}_{r_g}$, where $\underline{X}_{r_g} = \min(r_g)$ and $\overline{X}_{r_g} = \max(r_g)$. Therefore, (2) can be rewritten as $g(\underline{P}_i, P'_j) = m^3(a, \underline{X}_{r_g}, g(\underline{P}_i, \overline{P}_j))$, while (3) becomes $g(\overline{P}_i, P'_j) = m^3(a, g(\overline{P}_i, \underline{P}_j), \overline{X}_{r_g})$. It is immediate to see that $g(\overline{P}_i, P'_j) \geq g(\underline{P}_i, P'_j)$,

because (3) is at least a, whereas (2) is at most a. Hence, $c > g(\underline{P}_i, P'_j)$ because $c \geq b > a$. Moreover, $\tau|_{r_g}(\widetilde{P}_i) = c$ cannot be between $g(\overline{P}_i, P'_j)$ and $g(\underline{P}_i, P'_j)$. Otherwise, (1) would imply that $g(\widetilde{P}_i, P'_j) = c$, contradicting the initial hypothesis that $g(\widetilde{P}_i, P'_j) = \alpha$ (recall that $c \neq \alpha$ because $\beta \, \widetilde{P}_i \, \alpha$). Thus, $c > g(\overline{P}_i, P'_j) \geq g(\underline{P}_i, P'_j)$ and, therefore, $\alpha = g(\overline{P}_i, P'_j)$.

Take a preference P_i^α equal to P''_j. By Corollary 6, we have $g(P_i^\alpha, P'_j) = m^3(b, g(\underline{P}_i, P'_j), g(\overline{P}_i, P'_j))$.

- If $b \geq g(\overline{P}_i, P'_j)$, we have found a preference P_i^α satisfying properties (i) and (ii) above, since $g(P_i^\alpha, P'_j) = \alpha$ and, by definition, $\tau|_{r_f}(P_i^\alpha) = \tau|_{r_f}(P''_j) = w$.

- If, instead, $g(\overline{P}_i, P'_j) > b$, then $\alpha > b$. Moreover, $\alpha > a$ because $b > a$; and, by (3), $\alpha = m^3(a, g(\overline{P}_i, \underline{P}_j), \overline{X}_{r_g}) = g(\overline{P}_i, \underline{P}_j)$. Consider $g(\widetilde{P}_i, P''_j)$. By Corollary 6, $g(\widetilde{P}_i, P''_j) = m^3(b, g(\widetilde{P}_i, \underline{P}_j), g(\widetilde{P}_i, \overline{P}_j))$, where

$$g(\widetilde{P}_i, \underline{P}_j) = m^3(c, \underline{X}_{r_g}, g(\overline{P}_i, \underline{P}_j)) \tag{4}$$
$$g(\widetilde{P}_i, \overline{P}_j) = m^3(c, g(\underline{P}_i, \overline{P}_j), \overline{X}_{r_g}). \tag{5}$$

  Note that (4) can be rewritten as $g(\widetilde{P}_i, \underline{P}_j) = m^3(c, \underline{X}_{r_g}, \alpha) = \alpha$ because $c > \alpha$ (recall that $c > g(\overline{P}_i, P'_j) = \alpha$). Moreover, since (5) is at least equal to c and we have assumed above that $\alpha > b$, it follows that $g(\widetilde{P}_i, P''_j) = m^3(b, \alpha, g(\widetilde{P}_i, \overline{P}_j)) = \alpha$, contradicting the hypothesis that $g(\widetilde{P}_i, P''_j) = \beta \neq \alpha$.

Therefore, the previous argument shows that a preference P_i^α with the properties specified above exists in $\mathscr{S}$. In fact, it says that P_i^α can be taken equal to P''_j. Following a similar reasoning, it can be shown that a relation P_i^β with the properties stated in (iii) and (iv) also exists in $\mathscr{S}$. In fact, the desired preference can be obtained by setting P_i^β equal to P''_j.

Now, to complete the analysis, we proceed as in the proof of Lemma 5. First, recall that $w \neq \alpha$ and $w \neq \beta$ because $w \notin r_g$. Otherwise, we would have $\tau|_{r_g}(P'_j) = \tau|_{r_g}(P''_j)$. Moreover, $w \neq z$ because, by Corollary 5, $z \in r_g$.

Case 1. Suppose $\beta > w > \alpha$. If $\hat{P} \succ P'_j$, then $w \, P'_j \, \alpha$ implies $w \, \hat{P} \, \alpha$ (remember that $\hat{P}$ is the common preference relation of the profile $\hat{\mathbf{P}}_{-\{i,j\}}$). Define the sequence of profiles $\mathbf{P}^0_{-\{i,j\}} = (\hat{P}, \ldots, \hat{P})$, $\mathbf{P}^1_{-\{i,j\}} = (P^\alpha, \hat{P}, \ldots, \hat{P}), \ldots, \mathbf{P}^{n-2}_{-\{i,j\}} = (P^\alpha, \ldots, P^\alpha)$. For each $k = 0, \ldots, n-2$, let $x^k = f(P_i^\alpha, P'_j, \mathbf{P}^k_{-\{i,j\}})$. By SP of f, for each $k = 0, \ldots, n-3$, either $x^k \, \hat{P} \, x^{k+1}$ or $x^k = x^{k+1}$. Thus, $\alpha = x^0 \, \hat{P} \ldots x^k \, \hat{P} \ldots x^{n-2} = w$ (recall $\alpha \neq w$). Hence, by transitivity of $\hat{P}$, $\alpha \, \hat{P} \, w$, a contradiction.

In a similar way, if $P'_j \succ \hat{P}$, then $w \, P'_j \, \beta$ implies $w \, \hat{P} \, \beta$. Define the sequence of profiles $\mathbf{P}^0_{-\{i,j\}} = (\hat{P}, \ldots, \hat{P})$, $\mathbf{P}^1_{-\{i,j\}} = (P^\beta, \hat{P}, \ldots, \hat{P}), \ldots, \mathbf{P}^{n-2}_{-\{i,j\}} = (P^\beta, \ldots, P^\beta)$. For each $k = 0, \ldots, n-2$, let $y^k = f(P_i^\beta, P''_j, \mathbf{P}^k_{-\{i,j\}})$. By SP of f, for each $k = 0, \ldots, n-3$, either $y^k \, \hat{P} \, y^{k+1}$ or $y^k = y^{k+1}$. Thus, $\beta = y^0 \, \hat{P} \ldots y^k \, \hat{P} \ldots y^{n-2} = w$ (recall $\beta \neq w$). Therefore, by transitivity of $\hat{P}$, $\beta \, \hat{P} \, w$, a contradiction.

Case 2. Suppose that $\beta > \alpha > w$. The remaining case, where $w > \beta > \alpha$, is similar. If $P_j'' \succ \hat{P}$, then $w\,\hat{P}\,\alpha$, and we can use the argument of Case 1. Hence, assume that $\hat{P} \succ P_j''$. By SP, $\beta\, P_j''\, \alpha$. By $SC1$, $\beta\,\hat{P}\,\alpha$. If $\beta \in O^f_{-\{i,j\}}(P_i^\alpha, P_j')$, then there must be a $\bar{\bar{\mathbf{P}}}_{-\{i,j\}} \in \mathscr{S}^{n-2}$ such that $f(P_i^\alpha, P_j', \bar{\bar{\mathbf{P}}}_{-\{i,j\}}) = \beta$. Define the sequence of profiles

$$\mathbf{P}^0_{-\{i,j\}} = (\hat{P}, \dots, \hat{P}),\ \mathbf{P}^1_{-\{i,j\}} = (\bar{\bar{P}}_{\ell_1}, \hat{P}, \dots, \hat{P}),\ \dots,\ \mathbf{P}^{n-2}_{-\{i,j\}} = (\bar{\bar{P}}_{\ell_1}, \dots, \bar{\bar{P}}_{\ell_{n-2}}).$$

For each $k = 0, \dots, n-2$, let $x^k = f(P_i^\alpha, P_j', \mathbf{P}^k_{-\{i,j\}})$. By strategy-proofness, $\alpha = x^0\,\hat{P}\dots x^k\,\hat{P}\dots x^{n-2} = \beta$. By transitivity of $\hat{P}$, $\alpha\,\hat{P}\,\beta$, a contradiction. Therefore, $\beta \notin O^f_{-\{i,j\}}(P_i^\alpha, P_j')$.

By Corollary 5, $w \in O^f_{-\{i,j\}}(P_i^\alpha, P_j') \cap O^f_{-\{i,j\}}(P_i^\alpha, P_j'')$. Moreover, $\beta \in O^f_{-\{i,j\}}(P_i^\alpha, P_j'')$ because $f(P_i^\alpha, P_j'', \hat{\mathbf{P}}_{-\{i,j\}}) = \beta$ (recall that P_i^α and P_i^β are identical to P_j''). Consider any $P^\epsilon \in \mathscr{P}$ such that

(i) $\tau|_{O^f_{-\{i,j\}}(P_i^\alpha, P_j'')}(P^\epsilon) = \beta = \tau|_{O^f_{-\{i,j\}}(P_i^\alpha, P_j'')}(\hat{P})$

(ii) $\tau|_{O^f_{-\{i,j\}}(P_i^\alpha, P_j')}(P^\epsilon) = w = \tau|_{O^f_{-\{i,j\}}(P_i^\alpha, P_j')}(P_j'')$.

If $P^\epsilon \in \mathscr{S}$, then by Lemma 4, $f(P_i^\alpha, P_j'', \mathbf{P}^\epsilon_{-\{i,j\}}) = \beta$ and $f(P_i^\alpha, P_j', \mathbf{P}^\epsilon_{-\{i,j\}}) = w$. Therefore agent j can manipulate f at $(P_i^\alpha, P_j'', \mathbf{P}^\epsilon_{-\{i,j\}})$ via P_j', and we are done. If, on the contrary, $P^\epsilon \notin \mathscr{S}$, then the desired contradiction is found following the argument of Case 2 in the proof of Lemma 5.

Hence, by Case 1 and 2, we conclude that $O_i^f(P_j', \hat{\mathbf{P}}_{-\{i,j\}}) = O_i^f(P_j'', \hat{\mathbf{P}}_{-\{i,j\}})$. Applying a similar reasoning, we also have $O_i^f(P_j', \check{\mathbf{P}}_{-\{i,j\}}) = O_i^f(P_j'', \check{\mathbf{P}}_{-\{i,j\}})$.

Step 2. Next, we prove that $O_i^f(P_j', \hat{\mathbf{P}}_{-\{i,j\}}) = O_i^f(P_j'', \check{\mathbf{P}}_{-\{i,j\}})$. From Step 1, we know that $O_i^f(P_j'', \check{\mathbf{P}}_{-\{i,j\}}) = O_i^f(P_j', \check{\mathbf{P}}_{-\{i,j\}})$. Hence, it is enough to show that $O_i^f(P_j', \hat{\mathbf{P}}_{-\{i,j\}}) = O_i^f(P_j', \check{\mathbf{P}}_{-\{i,j\}})$. Define the sequence of profiles $\mathbf{P}^0_{-\{i,j\}} = (\hat{P}, \dots, \hat{P})$, $\mathbf{P}^1_{-\{i,j\}} = (\check{P}, \hat{P}, \dots, \hat{P})$, $\dots$, $\mathbf{P}^{n-2}_{-\{i,j\}} = (\check{P}, \dots, \check{P})$. To show that $O_i^f(P_j', \hat{\mathbf{P}}_{-\{i,j\}}) = O_i^f(P_j', \check{\mathbf{P}}_{-\{i,j\}})$, it is enough to prove that for all $k = 1, \dots, n-2$, $O_i^f(P_j', \mathbf{P}^{k-1}_{-\{i,j\}}) = O_i^f(P_j', \mathbf{P}^k_{-\{i,j\}})$. Suppose, by contradiction, there exists $1 \le k^* \le n-2$ such that

$$O_i^f(P_j', \mathbf{P}^{k^*-1}_{-\{i,j\}}) \neq O_i^f(P_j', \mathbf{P}^{k^*}_{-\{i,j\}}). \tag{6}$$

Recall that

$$\mathbf{P}^{k^*-1}_{-\{i,j\}} = (\underbrace{\check{P}, \dots, \check{P}}_{k^*-1}, \underbrace{\hat{P}, \dots, \hat{P}}_{n-k^*-1}) \quad \text{and} \quad \mathbf{P}^{k^*}_{-\{i,j\}} = (\underbrace{\check{P}, \dots, \check{P}, \check{P}}_{k^*}, \underbrace{\hat{P}, \dots, \hat{P}}_{n-k^*-2}).$$

That is, the profiles $\mathbf{P}^{k^*-1}_{-\{i,j\}}$ and $\mathbf{P}^{k^*}_{-\{i,j\}}$ differ only in one preference relation. Abusing the notation, assume this ordering corresponds to agent k^*. Then, (6) can be rewritten as

$$O_i^f(P_j', \hat{P}_{k^*}, \mathbf{P}^{k^*-1}_{-\{i,j,k^*\}}) \neq O_i^f(P_j', \check{P}_{k^*}, \mathbf{P}^{k^*}_{-\{i,j,k^*\}}). \tag{7}$$

Fix

$$\mathbf{P}^{k^*-1}_{-\{i,j,k^*\}} = (\underbrace{\check{P},\dots,\check{P}}_{k^*-1}, \underbrace{\hat{P},\dots,\hat{P}}_{n-k^*-2}) = \mathbf{P}^{k^*}_{-\{i,j,k^*\}}.$$

Define the social choice rule $g : \mathscr{S}^3 \to X$ in such a way that for all $(P_i, P_j, P_{k^*}) \in \mathscr{S}^3$, $g(P_i, P_j, P_{k^*}) = f(P_i, P_j, P_{k^*}, \mathbf{P}^{k^*-1}_{-\{i,j,k^*\}})$. By Corollary 5, $z \in r_g = O^f_{\{i,j,k^*\}}(\mathbf{P}^{k^*-1}_{-\{i,j,k^*\}})$. Hence, $\tau|_{r_g}(\hat{P}_{k^*}) = \tau|_{r_g}(\check{P}_{k^*}) = z$ because, by hypothesis, $\tau|_{r_f}(\hat{P}_{k^*}) = \tau|_{r_f}(\check{P}_{k^*}) = z$. Since g is SP, by Step 1, $O^g_i(P'_j, \hat{P}_{k^*}) = O^g_i(P'_j, \check{P}_{k^*})$, which contradicts (7). Therefore, $O^f_i(P'_j, \hat{\mathbf{P}}_{-\{i,j\}}) = O^f_i(P''_j, \check{\mathbf{P}}_{-\{i,j\}})$.

Step 3. Suppose now that for some $K \subset N \setminus \{i\}$ and any $\mathbf{P}'_K, \mathbf{P}''_K \in \mathscr{S}^K$ with the property that $\forall j \in K$, $\tau|_{r_f}(P'_j) = \tau|_{r_f}(P''_j)$,

$$O^f_i(\mathbf{P}'_K, \hat{\mathbf{P}}_{\bar{K}\setminus\{i\}}) = O^f_i(\mathbf{P}''_K, \check{\mathbf{P}}_{\bar{K}\setminus\{i\}}). \tag{8}$$

Notice that Step 2 deals with the particular case where $K = \{j\}$. Fix any $h \in \bar{K} \setminus \{i\}$ and any two preferences $P'_h, P''_h \in \mathscr{S}$ for which $\tau|_{r_f}(P'_h) = \tau|_{r_f}(P''_h)$. We want to show that

$$O^f_i(\mathbf{P}'_{K\cup\{h\}}, \hat{\mathbf{P}}_{\bar{K}\setminus\{i,h\}}) = O^f_i(\mathbf{P}''_{K\cup\{h\}}, \check{\mathbf{P}}_{\bar{K}\setminus\{i,h\}}). \tag{9}$$

To fix the notation, let $K = \{\ell_1, \dots, \ell_{|K|}\}$. Define the sequence of profiles

$$\begin{aligned}
\mathbf{P}^0_{K\cup\{h\}} &= (P'_{\ell_1}, \dots, P'_{\ell_{|K|}}, P'_h) = \mathbf{P}'_{K\cup\{h\}} \\
\mathbf{P}^1_{K\cup\{h\}} &= (P''_{\ell_1}, P'_{\ell_2}, \dots, P'_{\ell_{|K|}}, P'_h) \\
&\vdots \\
\mathbf{P}^{|K|+1}_{K\cup\{h\}} &= (P''_{\ell_1}, \dots, P''_{\ell_{|K|}}, P''_h) = \mathbf{P}''_{K\cup\{h\}}.
\end{aligned}$$

We argue next that the proof of (9) can be reduced to showing that

$$\forall t = 1, \dots, |K|+1,\ O^f_i(\mathbf{P}^{t-1}_{K\cup\{h\}}, \hat{\mathbf{P}}_{\bar{K}\setminus\{i,h\}}) = O^f_i(\mathbf{P}^{t}_{K\cup\{h\}}, \hat{\mathbf{P}}_{\bar{K}\setminus\{i,h\}}). \tag{10}$$

In effect, if (10) holds, then $O^f_i(\mathbf{P}'_{K\cup\{h\}}, \hat{\mathbf{P}}_{\bar{K}\setminus\{i,h\}}) = O^f_i(\mathbf{P}''_{K\cup\{h\}}, \hat{\mathbf{P}}_{\bar{K}\setminus\{i,h\}})$. Using a similar reasoning, it also follows that $O^f_i(\mathbf{P}'_{K\cup\{h\}}, \check{\mathbf{P}}_{\bar{K}\setminus\{i,h\}}) = O^f_i(\mathbf{P}''_{K\cup\{h\}}, \check{\mathbf{P}}_{\bar{K}\setminus\{i,h\}})$. Therefore, in order to prove (9), it would be enough to show that

$$O^f_i(\mathbf{P}'_{K\cup\{h\}}, \hat{\mathbf{P}}_{\bar{K}\setminus\{i,h\}}) = O^f_i(\mathbf{P}'_{K\cup\{h\}}, \check{\mathbf{P}}_{\bar{K}\setminus\{i,h\}}). \tag{11}$$

Following the argument of Step 2, consider a sequence of profiles

$$\begin{aligned}
\mathbf{P}^0_{\bar{K}\setminus\{i,h\}} &= (\hat{P}, \dots, \hat{P}) = \hat{\mathbf{P}}_{\bar{K}\setminus\{i,h\}} \\
\mathbf{P}^1_{\bar{K}\setminus\{i,h\}} &= (\check{P}, \hat{P}, \dots, \hat{P}) \\
&\vdots \\
\mathbf{P}^{|\bar{K}|-2}_{\bar{K}\setminus\{i,h\}} &= (\check{P}, \dots, \check{P}) = \check{\mathbf{P}}_{\bar{K}\setminus\{i,h\}}.
\end{aligned}$$

To show (11), it is sufficient to prove that for each $t = 1, \dots, |\bar{K}| - 2$,

$$O_i^f(\mathbf{P}'_{K\cup\{h\}}, \mathbf{P}^{t-1}_{\bar{K}\setminus\{i,h\}}) = O_i^f(\mathbf{P}'_{K\cup\{h\}}, \mathbf{P}^{t}_{\bar{K}\setminus\{i,h\}}).$$

Notice that $\mathbf{P}^{t-1}_{\bar{K}\setminus\{i,h\}}$ and $\mathbf{P}^{t}_{\bar{K}\setminus\{i,h\}}$ differ only in one preference relation. Without loss of generality, assume that they differ in agent ℓ's ordering. Fix $\mathbf{P}^{t-1}_{\bar{K}\setminus\{i,h,\ell\}}$ $(= \mathbf{P}^{t}_{\bar{K}\setminus\{i,h,\ell\}})$. Define the social choice rule $g : \mathscr{S}^{|K|+3} \to X$ in such a way that for every $\mathbf{P}_{K\cup\{i,h,\ell\}} \in \mathscr{S}^{|K|+3}$, $g(\mathbf{P}_{K\cup\{i,h,\ell\}}) = f(\mathbf{P}_{K\cup\{i,h,\ell\}}, \mathbf{P}^{t-1}_{\bar{K}\setminus\{i,h,\ell\}})$. By Corollary 5, $z \in r_g = O^f_{K\cup\{i,h,\ell\}}(\mathbf{P}^{t-1}_{\bar{K}\setminus\{i,h,\ell\}})$ (recall that $\tau|_{r_f}(\hat{P}) = \tau|_{r_f}(\check{P}) = z$). Hence, since g is SP and $\tau|_{r_g}(\hat{P}_\ell) = \tau|_{r_g}(\check{P}_\ell)$, using the argument behind (10) (with the proviso that the roles of h and ℓ are interchanged, so that $\bar{K} \setminus \{i,\ell\} = \{h\}$), we have that $O_i^g(\mathbf{P}'_K, \hat{P}_\ell, P'_h) = O_i^g(\mathbf{P}'_K, \check{P}_\ell, P'_h)$,[20] and we are done: i.e., (11) follows from the definition of g.

So, as we said above, Step 3 requires only that (10) be valid. That is, we need to prove that for each $t = 1, \dots, |K| + 1$,

$$O_i^f(\mathbf{P}^{t-1}_{K\cup\{h\}}, \hat{\mathbf{P}}_{\bar{K}\setminus\{i,h\}}) = O_i^f(\mathbf{P}^{t}_{K\cup\{h\}}, \hat{\mathbf{P}}_{\bar{K}\setminus\{i,h\}}).$$

Once again, since $\mathbf{P}^{t-1}_{K\cup\{h\}}$ and $\mathbf{P}^{t}_{K\cup\{h\}}$ differ only in one preference relation, without loss of generality and to simplify the notation, assume $\mathbf{P}^{t-1}_{K\cup\{h\}} = (P''_{\ell_1}, \dots, P''_{\ell_{|K|}}, P'_h)$ and $\mathbf{P}^{t}_{K\cup\{h\}} = (P''_{\ell_1}, \dots, P''_{\ell_{|K|}}, P''_h)$.

Fix $\hat{\mathbf{P}}_{\bar{K}\setminus\{i,h\}}$. Define the social choice rule $g_1 : \mathscr{S}^{|K|+2} \to X$ in such a way that for all $\mathbf{P}_{K\cup\{i,h\}} \in \mathscr{S}^{|K|+2}$, $g_1(\mathbf{P}_{K\cup\{i,h\}}) = f(\mathbf{P}_{K\cup\{i,h\}}, \hat{\mathbf{P}}_{\bar{K}\setminus\{i,h\}})$. Since g_1 is SP, if $\tau|_{r_{g_1}}(P''_h) = \tau|_{r_{g_1}}(P'_h)$, then we can use the argument behind (8) to conclude that $O_i^{g_1}(P''_{\ell_1}, \dots, P''_{\ell_{|K|}}, P'_h) = O_i^{g_1}(P''_{\ell_1}, \dots, P''_{\ell_{|K|}}, P''_h)$, and we are done.

If, on the contrary, $\tau|_{r_{g_1}}(P''_h) \neq \tau|_{r_{g_1}}(P'_h)$, then fix $(P''_{\ell_1}, \dots, P''_{\ell_{|K|}}) \equiv \mathbf{P}''_K$ and define the social choice rule $g_2 : \mathscr{S}^2 \to X$ such that for each $(P_i, P_h) \in \mathscr{S}^2$, $g_2(P_i, P_h) = g_1(P_i, P_h, \mathbf{P}''_K)$. Obviously, g_2 is SP with $r_{g_2} = O^{g_1}_{\{i,h\}}(\mathbf{P}''_K)$. If $\tau|_{r_{g_2}}(P''_h) = \tau|_{r_{g_2}}(P'_h)$, then by Lemma 6, $O_i^{g_2}(P'_h) = O_i^{g_2}(P''_h)$, and we are done. Thus, suppose that $\tau|_{r_{g_2}}(P''_h) = b > a = \tau|_{r_{g_2}}(P'_h)$ for some $a, b \in r_{g_2}$, implying that $P''_h \succ P'_h$. The desired result is then obtained following the reasoning of Step 1.

Specifically, assume by way of contradiction that $O_i^{g_2}(P'_h) \neq O_i^{g_2}(P''_h)$. Without loss of generality, suppose there exists $\alpha \in r_{g_2}$ such that $\alpha \in O_i^{g_2}(P'_h)$ and $\alpha \notin O_i^{g_2}(P''_h)$. Hence, there must be a $\widetilde{P}_i \in \mathscr{S}$ such that $g_2(\widetilde{P}_i, P'_h) = \alpha \neq \beta = g_2(\widetilde{P}_i, P''_h)$ for some $\beta \in r_{g_2}$. By SP, $\alpha\, P'_h\, \beta$ and $\beta\, P''_h\, \alpha$. By single-crossing, $P''_h \succ P'_h$ implies $\beta > \alpha$.

Let $\tau|_{r_f}(P'_h) = w_h$. By hypothesis, $\tau|_{r_f}(P''_h) = w_h$. Following the argument of Step 1 (i.e., exploiting that g_2 is a median function), there exist $P_i^\alpha, P_i^\beta \in \mathscr{S}$ such that (i) $\tau|_{r_f}(P_i^\alpha) = w_h$, (ii) $g_2(P_i^\alpha, P'_h) = \alpha$, (iii) $\tau|_{r_f}(P_i^\beta) = w_h$, and (iv) $g_2(P_i^\beta, P''_h) = \beta$. Actually, P_i^α and P_i^β can be taken to be identical to each other.

Next, fix any $\ell \in K = \{\ell_1, \dots, \ell_{|K|}\}$ and $(\widetilde{P}_i, \mathbf{P}''_{K\setminus\{\ell\}})$. Define the social choice rule $g^\ell : \mathscr{S}^2 \to X$ in such a way that for all $(P_\ell, P_h) \in \mathscr{S}^2$, $g^\ell(P_\ell, P_h) = g_1(\widetilde{P}_i, P_h, P_\ell, \mathbf{P}''_{K\setminus\{\ell\}})$. Note that

[20] Bear in mind that, by construction, in (10) the profiles $\mathbf{P}^{t-1}_{K\cup\{h\}}$ and $\mathbf{P}^{t}_{K\cup\{h\}}$ differ only with respect to a single preference relation.

g^ℓ is SP; and, by the definition of g^ℓ, $g^\ell(P''_\ell, P'_h) = \alpha$ and $g^\ell(P''_\ell, P''_h) = \beta$. Thus, since by Lemma 5, g^ℓ is TO, $\tau|_{r_{g^\ell}}(P''_h) \neq \tau|_{r_{g^\ell}}(P'_h)$. Once again, following the argument of Step 1, there exist $P^\alpha_\ell, P^\beta_\ell \in \mathscr{S}$ such that (i) $\tau|_{r_f}(P^\alpha_\ell) = w_h$, (ii) $g^\ell(P^\alpha_\ell, P'_h) = \alpha$, (iii) $\tau|_{r_f}(P^\beta_\ell) = w_h$, and (iv) $g^\ell(P^\beta_\ell, P''_h) = \beta$. Furthermore, P^α_ℓ and P^β_ℓ are identical.

Since $\ell \in K$ has been arbitrarily chosen, we conclude from the previous paragraph that there must exist $P^\alpha_{\ell_1}, \dots, P^\alpha_{\ell_{|K|}} \in \mathscr{S}$ and $P^\beta_{\ell_1}, \dots, P^\beta_{\ell_{|K|}} \in \mathscr{S}$ such that, for all $s = \ell_1, \dots, \ell_{|K|}$, $\tau|_{r_f}(P^\alpha_s) = \tau|_{r_f}(P^\beta_s) = w_h$, $f(P^\alpha_i, P'_h, \mathbf{P}^\alpha_K, \hat{\mathbf{P}}_{\bar{K}\setminus\{i,h\}}) = \alpha$, and $f(P^\beta_i, P''_h, \mathbf{P}^\beta_K, \hat{\mathbf{P}}_{\bar{K}\setminus\{i,h\}}) = \beta$, where $\mathbf{P}^\alpha_K = (P^\alpha_{\ell_1}, \dots, P^\alpha_{\ell_{|K|}})$ and $\mathbf{P}^\beta_K = (P^\beta_{\ell_1}, \dots, P^\beta_{\ell_{|K|}})$.

Notice that $w_h \neq \alpha$ because $\alpha \in r_{g_2}$ and $w_h \notin r_{g_2}$. Similarly, $\beta \neq w_h$ and $z \neq w_h$ (recall that $\tau|_{r_{g_2}}(P''_h) \neq \tau|_{r_{g_2}}(P'_h)$ and $\tau|_{r_{g_1}}(P''_h) \neq \tau|_{r_{g_1}}(P'_h)$). Thus, to complete the analysis we repeat the argument of Cases 1 and 2 in Step 1. That is, if for example $\beta > w_h > \alpha$ and $\hat{P} \succ P'_h$, then $w_h \, P'_h \, \alpha$ implies $w_h \, \hat{P} \, \alpha$. Define the sequence of profiles

$$\begin{aligned}
\mathbf{P}^0_{\bar{K}\setminus\{i,h\}} &= (\hat{P}, \dots, \hat{P}) \\
\mathbf{P}^1_{\bar{K}\setminus\{i,h\}} &= (P^\alpha, \hat{P}, \dots, \hat{P}) \\
&\vdots \\
\mathbf{P}^{|\bar{K}|-2}_{\bar{K}\setminus\{i,h\}} &= (P^\alpha, \dots, P^\alpha).
\end{aligned}$$

where P^α is set identical to P^α_i. For each $s = 0, \dots, |\bar{K}|-2$, let $x^s = f(P^\alpha_i, P'_h, \mathbf{P}^\alpha_K, \mathbf{P}^s_{\bar{K}\setminus\{i,h\}})$. By SP of f, for any $s = 0, \dots, |\bar{K}|-3$, either $x^s = x^{s+1}$ or, if $x^s \neq x^{s+1}$, then $x^s \, \hat{P} \, x^{s+1}$. Hence, since $w_h \neq \alpha$, we have $\alpha = x^0 \, \hat{P} \dots x^s \, \hat{P} \dots x^{|\bar{K}|-2} = w_h$, and, by transitivity of $\hat{P}$, $\alpha \, \hat{P} \, w_h$, a contradiction. (The remaining cases are solved in a similar fashion.) Therefore, (10) holds.

Finally, by Step 2, (8) is satisfied for $K = \{j\}$ for all $j \in N \setminus \{i\}$. By the induction argument, (8) holds for *any* $K \subset N \setminus \{i\}$. In particular, by setting $K = N \setminus \{i, k\}$, for any $k \neq i$, we have that $O^f_i(P'_{N\setminus\{i,k\}}, \hat{P}_k) = O^f_i(P''_{N\setminus\{i,k\}}, \check{P}_k)$. Therefore, since i, k, $P'_{N\setminus\{i,k\}}$, $\hat{P}_k$, $P''_{N\setminus\{i,k\}}$, and $\check{P}_k$ were arbitrarily chosen, for each $i \in N$ and every pair of profiles $\mathbf{P}'_{-i}, \mathbf{P}''_{-i} \in \mathscr{S}^{n-1}$ with the property that $\tau|_{r_f}(P'_j) = \tau|_{r_f}(P''_j)$ for all $j \in N \setminus \{i\}$, $O^f_i(\mathbf{P}'_{-i}) = O^f_i(\mathbf{P}''_{-i})$. □

We are now ready to prove Proposition 2.

Proof of Proposition 2. Suppose, by contradiction, there exist $i \in N$, $(P'_i, \mathbf{P}'_{-i}) \in \mathscr{S}^n$, and $P''_i \in \mathscr{S}$ such that $\tau|_{r_f}(P'_i) = \tau|_{r_f}(P''_i)$ and $f(P'_i, \mathbf{P}'_{-i}) = x \neq y = f(P''_i, \mathbf{P}'_{-i})$. Fix any $j \neq i$. Since preferences are strict, $x \neq y$ implies that either $x \, P'_j \, y$ or $y \, P'_j \, x$. Without loss of generality, assume that $y \, P'_j \, x$. By Lemma 7, $O^f_j(P'_i, \mathbf{P}'_{-\{i,j\}}) = O^f_j(P''_i, \mathbf{P}'_{-\{i,j\}})$. Thus, $y \in O^f_j(P'_i, \mathbf{P}'_{-\{i,j\}})$. That is, $\exists \widehat{P}_j \in \mathscr{S}$ such that $f(\widehat{P}_j, P'_i, \mathbf{P}'_{-\{i,j\}}) = y$. However, since $y \, P'_j \, x$, this means that j can manipulate f at $(P'_i, \mathbf{P}'_{-i})$ via $\widehat{P}_j$, a contradiction. Hence, f is TO. □

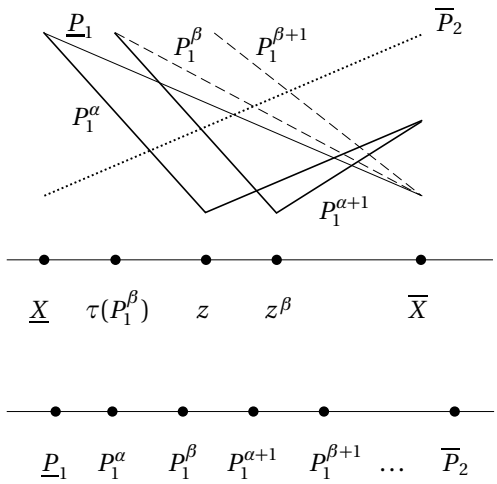


FIGURE 5. Fixed ballots over X.

C. PROOF OF THEOREM 1

The sufficiency is immediate from Proposition 1 and Definition 9. To show the necessity, suppose that f is UN, AN and SP on $\mathscr{S}^n$. By Proposition 2, f is also TO on $\mathscr{S}^n$.

Consider first the case where $|N| = 2$. Fix any profile $\mathbf{P} \in \mathscr{S}^2$. By Lemma 1, $f(P_1, P_2) = m^3(\tau(P_1), f(\underline{P}_1, P_2), f(\overline{P}_1, P_2))$. Applying Lemma 1 once again, $f(\underline{P}_1, P_2) = m^3(\tau(P_2), f(\underline{P}_1, \underline{P}_2), f(\underline{P}_1, \overline{P}_2))$ and $f(\overline{P}_1, P_2) = m^3(\tau(P_2), f(\overline{P}_1, \underline{P}_2), f(\overline{P}_1, \overline{P}_2))$. By unanimity, $f(\underline{P}_1, \underline{P}_2) = \underline{X}$ and $f(\overline{P}_1, \overline{P}_2) = \overline{X}$. By anonymity, $f(\underline{P}_1, \overline{P}_2) = f(\overline{P}_1, \underline{P}_2)$.

Next, we show that $f(\underline{P}_1, \overline{P}_2) \in \{\tau(P) \in X : P \in \mathscr{S}\}$. Suppose not. Without loss of generality, assume that $f(\underline{P}_1, \overline{P}_2) = z \neq \tau(P)$ for all $P \in \mathscr{S}$. Consider a preference $P_1^\alpha \in \mathscr{P}$ with the property that $\tau(P_1^\alpha) = \tau(\underline{P}_1)$ and $\overline{X}\, P_1^\alpha\, z$. If $P_1^\alpha \in \mathscr{S}$, we are done. By TO, $f(P_1^\alpha, \overline{P}_2) = z$, and agent 1 can manipulate f at $(P_1^\alpha, \overline{P}_2)$ via $\overline{P}_1$.

If, instead, $P_1^\alpha \notin \mathscr{S}$, there must exist a $P^* \in \mathscr{S}$ such that $\tau(P^*) > \tau(\underline{P}_1)$ and $z\, P^*\, \overline{X}$. Let $P_1^\beta = \min_{\succ}\{P' \in \mathscr{S} : \tau(P') > \underline{X}\}$. Clearly, $z\, P_1^\beta\, \overline{X}$ because either P_1^β coincides with P^* or $P_1^\beta \prec P^*$. Let $f(P_1^\beta, \overline{P}_2) = z^\beta$. If $z > z^\beta$, agent 1 would manipulate f at $(\underline{P}_1, \overline{P}_2)$ via P_1^β. Similarly, if $z^\beta = \overline{X}$, then 1 would manipulate f at $(P_1^\beta, \overline{P}_2)$ via $\underline{P}_1$. Hence, $\overline{X} > z^\beta \geq z$.

Suppose $z^\beta = \tau(P_1^\beta)$. Then, $z^\beta > z$. Furthermore, there exists a $P_1^{\alpha'} \in \mathscr{S}$ such that $\tau(P_1^{\alpha'}) = \underline{X}$ and $z^\beta\, P_1^{\alpha'}\, z$. Indeed, to rule out $P_1^{\alpha'}$ from $\mathscr{S}$ there should be a $P^{**} \in \mathscr{S}$ such that $\tau(P^{**}) > \underline{X}$ and $z\, P^{**}\, z^\beta$. By the definition of P_1^β, $P^{**} \succ P_1^\beta$ (note that they cannot be equal because by hypothesis $z^\beta = \tau(P_1^\beta)$ and $z\, P^{**}\, z^\beta$), and, by $SC2$, we would have $z\, P_1^\beta\, z^\beta$, a contradiction. Thus, $P_1^{\alpha'} \in \mathscr{S}$. By TO, $f(P_1^{\alpha'}, \overline{P}_2) = z$. Hence agent 1 can manipulate f at $(P_1^{\alpha'}, \overline{P}_2)$ via P_1^β, a contradiction. Therefore, $z^\beta \neq \tau(P_1^\beta)$.

Consider a preference $P_1^{\alpha+1} \in \mathscr{P}$ such that $\tau(P_1^{\alpha+1}) = \tau(P_1^\beta)$ and $\overline{X}\, P_1^{\alpha+1}\, z^\beta$ (see Figure 5). If $P_1^{\alpha+1} \in \mathscr{S}$, by TO, $f(P_1^{\alpha+1}, \overline{P}_2) = z^\beta$. Thus, agent 1 can manipulate f at $(P_1^{\alpha+1}, \overline{P}_2)$ via $\overline{P}_1$. If, on the contrary, $P_1^{\alpha+1} \notin \mathscr{S}$, then we can repeat the previous argument and

find a preference $P_1^{\beta+1} \in \mathscr{S}$ such that $\tau(P_1^{\beta+1}) > \tau(P_1^{\beta})$ and $z^{\beta}\, P_1^{\beta+1}\, \overline{X}$. Since X is finite and in each step the top of the blocking ordering becomes larger, the sequence $\tau(P_1^{\beta}), \tau(P_1^{\beta+1}), \ldots$ approaches $\tau(\overline{P}_1)$. Therefore, if we continue applying the same argument repeatedly, at some point we will either find the desired contradiction, or a preference $P_1^{\beta+\ell} \in \mathscr{S}$ such that (i) $\tau(P_1^{\beta+\ell}) = \tau(\overline{P}_1)$ and (ii) $z^{\beta+\ell-1}\, P_1^{\beta+\ell}\, \overline{X}$, which leads to a violation of SP (because $f(P_1^{\beta+\ell}, \overline{P}_2) = \overline{X}$). Hence, $f(\underline{P}_1, \overline{P}_2) \in \{\tau(P) \in X : P \in \mathscr{S}\}$.

Let $f(\underline{P}_1, \overline{P}_2) = \tau(P)$ for some $P \in \mathscr{S}$. Then, from Lemma 1, either $f(\underline{P}_1, P_2) = \tau(P)$ and $f(\overline{P}_1, P_2) = \tau(P_2)$ if $P_2 \succ P$ or, otherwise, $f(\underline{P}_1, P_2) = \tau(P_2)$ and $f(\overline{P}_1, P_2) = \tau(P)$. Therefore, $f(P_1, P_2) = m^3(\tau(P_1), \tau(P_2), \tau(P))$. Thus, since $(P_1, P_2) \in \mathscr{S}^2$ was arbitrarily chosen and $f(\underline{P}_1, \overline{P}_2)$ is independent of (P_1, P_2), we conclude that $f \in PR$.

Let us now extend the proof for $|N| = n > 2$. For all $K \subseteq N$, let $a_{|K|} = f(\underline{\mathbf{P}}_K, \overline{\mathbf{P}}_{\bar{K}})$, where $\bar{K} = N \setminus K$. By UN, $K = \varnothing$ implies $a_0 = f(\overline{P}_1, \ldots, \overline{P}_n) = \overline{X}$. Similarly, $K = N$ implies $a_n = f(\underline{P}_1, \ldots, \underline{P}_n) = \underline{X}$. By AN, $a_1 = f(\underline{P}_i, \overline{\mathbf{P}}_{-i}), \forall \{i\} \subset N$; $a_2 = f(\underline{\mathbf{P}}_{\{i,j\}}, \overline{\mathbf{P}}_{-\{i,j\}}) \forall \{i,j\} \subseteq N; \ldots$; and, $a_{n-1} = f(\underline{\mathbf{P}}_{-j}, \overline{P}_j), \forall \{j\} \subset N$. By TM, $a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n$.

Next, notice that $a_k \in \{\tau(P) \in X : P \in \mathscr{S}\}$ for all $k = 0, 1, \ldots, n$. In effect, if $k = 0$ or $k = n$, the result follows from UN. Assume that $a_k = f(\underline{P}_1, \ldots, \underline{P}_k, \overline{P}_{k+1}, \ldots, \overline{P}_n) \in \{\tau(P) \in X : P \in \mathscr{S}\}$ for some $k = 0, 1, \ldots, n-2$ and let us prove the claim for a_{k+1}. On the contrary, suppose that $a_{k+1} = f(\underline{P}_1, \ldots, \underline{P}_{k+1}, \overline{P}_{k+2}, \ldots, \overline{P}_n) \neq \tau(P)$ for all $P \in \mathscr{S}$. Then, $a_k > a_{k+1}$. Following the argument illustrated in Figure 5 for $|N| = 2$, there exist $P_{k+1}^{\alpha} \in \mathscr{S}$ and $P_{k+1}^{\beta} \in \mathscr{S}$ such that $\tau(P_{k+1}^{\alpha}) = \tau(P_{k+1}^{\beta})$ and $a_k\, P_{k+1}^{\alpha}\, f(\underline{P}_1, \ldots, \underline{P}_k, P_{k+1}^{\beta}, \overline{P}_{k+2}, \ldots, \overline{P}_n)$. By TO, $f(\underline{P}_1, \ldots, \underline{P}_k, P_{k+1}^{\alpha}, \overline{P}_{k+2}, \ldots, \overline{P}_n) = f(\underline{P}_1, \ldots, \underline{P}_k, P_{k+1}^{\beta}, \overline{P}_{k+2}, \ldots, \overline{P}_n)$. Hence, agent $k+1$ can manipulate f at $(\underline{P}_1, \ldots, \underline{P}_k, P_{k+1}^{\alpha}, \overline{P}_{k+2}, \ldots, \overline{P}_n)$ via $\overline{P}_{k+1}$ (which results in a_k being chosen), a contradiction. Therefore, $a_{k+1} \in \{\tau(P) \in X : P \in \mathscr{S}\}$.

Now, fix any profile $\mathbf{P} = (P_1, \ldots, P_n) \in \mathscr{S}^n$ and relabel N if necessary so that $\tau(P_n) \geq \tau(P_{n-1}) \geq \cdots \geq \tau(P_1)$. Exactly one of the following must hold:

$$\begin{array}{ccccc}
\tau(P_1) & \geq & a_0 & & \\
a_0 & > & \tau(P_1) & > & a_1 \\
\tau(P_2) & \geq & a_1 & \geq & \tau(P_1) \\
a_1 & > & \tau(P_2) & > & a_2 \\
 & & \cdots & & \\
\tau(P_n) & \geq & a_{n-1} & \geq & \tau(P_{n-1}) \\
a_{n-1} & > & \tau(P_n) & > & a_n \\
a_n & \geq & \tau(P_n). & &
\end{array}$$

There are three cases to analyze.

Case 1: either $\tau(P_1) \geq a_0$ or $a_n \geq \tau(P_n)$. Without loss of generality, suppose $\tau(P_1) \geq a_0$. Then, $\forall i \in N$, $\tau(P_i) = \overline{X}$. By UN, $f(P_1, \ldots, P_n) = \overline{X} = m^{2n-1}(\tau(P_1), \ldots, \tau(P_n), a_1, \ldots, a_{n-1})$ as desired.

Case 2: $a_{j-1} > \tau(P_j) > a_j$ for some $j \in N$. Note that

$$\tau(P_j) = m^{2n-1}(\tau(P_1), \ldots, \tau(P_{j-1}), \tau(P_j), \tau(P_{j+1}), \ldots, \tau(P_n), a_1, \ldots, a_{j-1}, a_j, \ldots, a_{n-1}),$$

because the elements $\tau(P_1), \ldots, \tau(P_{j-1}), a_j, \ldots, a_{n-1}$ are smaller than or equal to $\tau(P_j)$. Assume, by way of contradiction, that $f(P_1, \ldots, P_n) \neq \tau(P_j)$. Without loss of

generality, suppose $f(P_1,\dots,P_n) > \tau(P_j)$. By TM, $f(P_1,\dots,P_{n-1},\overline{P}_n) \geq f(P_1,\dots,P_n)$; hence $f(P_1,\dots,P_{n-1},\overline{P}_n) > \tau(P_j)$. Repeating the step for all $i > j$ eventually yields $f(P_1,\dots,P_j,\overline{P}_{j+1},\dots,\overline{P}_n) > \tau(P_j)$. By Lemma 1,

$$\begin{aligned} &f(P_1,\dots,P_{j-1},P_j,\overline{P}_{j+1},\dots,\overline{P}_n) \\ &\qquad = m^3(\tau(P_j), f(P_1,\dots,P_{j-1},\underline{P}_j,\overline{P}_{j+1},\dots,\overline{P}_n), f(P_1,\dots,P_{j-1},\overline{P}_j,\overline{P}_{j+1},\dots,\overline{P}_n)). \end{aligned}$$

Therefore,

$$f(P_1,\dots,P_{j-1},P_j,\overline{P}_{j+1},\dots,\overline{P}_n) = f(P_1,\dots,P_{j-1},\underline{P}_j,\overline{P}_{j+1},\dots,\overline{P}_n),$$

because by TM, $f(P_1,\dots,P_{j-1},\overline{P}_j,\overline{P}_{j+1},\dots,\overline{P}_n) \geq f(P_1,\dots,P_{j-1},\underline{P}_j,\overline{P}_{j+1},\dots,\overline{P}_n)$ and, by hypothesis, $f(P_1,\dots,P_j,\overline{P}_{j+1},\dots,\overline{P}_n) > \tau(P_j)$. Thus, repeating the argument for all $i = 1,\dots,j-1$, we get

$$f(\underline{P}_1,\dots,\underline{P}_j,\overline{P}_{j+1},\dots,\overline{P}_n) > \tau(P_j),$$

which contradicts the initial hypothesis that $\tau(P_j) > a_j$. Therefore, $f(\mathbf{P}) = \tau(P_j) = m^{2n-1}(\tau(P_1),\dots,\tau(P_n),a_1,\dots,a_{n-1})$, as required.

Case 3: $\tau(P_j) \geq a_{j-1} \geq \tau(P_{j-1})$ for some $j = 2,\dots,n$. Note that

$$a_{j-1} = m^{2n-1}(\tau(P_1),\dots,\tau(P_{j-1}),\tau(P_j),\dots,\tau(P_n),a_1,\dots,a_{j-1},a_j,\dots,a_{n-1}),$$

because elements $\tau(P_1),\dots,\tau(P_{j-1}),a_j,\dots,a_{n-1}$ are smaller than or equal to a_{j-1}. Suppose, by way of contradiction, that $f(P_1,\dots,P_n) \neq a_{j-1}$. Without loss of generality, assume $f(P_1,\dots,P_n) > a_{j-1}$. By TM, $f(P_1,\dots,P_{n-1},\overline{P}_n) > a_{j-1}$. Repeating the step for all $i \geq j$ eventually yields $f(P_1,\dots,P_{j-1},\overline{P}_j,\dots,\overline{P}_n) > a_{j-1}$, which implies that

$$f(P_1,\dots,P_{j-1},\overline{P}_j,\dots,\overline{P}_n) > \tau(P_{j-1}), \tag{12}$$

because by hypothesis $a_{j-1} \geq \tau(P_{j-1})$. By Lemma 1, $f(P_1,\dots,P_{j-1},\overline{P}_j,\dots,\overline{P}_n) = m^3(\tau(P_{j-1}), f(P_1,\dots,P_{j-2},\underline{P}_{j-1},\overline{P}_j,\dots,\overline{P}_n), f(P_1,\dots,P_{j-2},\overline{P}_{j-1},\overline{P}_j,\dots,\overline{P}_n))$. Therefore,

$$f(P_1,\dots,P_{j-2},P_{j-1},\overline{P}_j,\dots,\overline{P}_n) = f(P_1,\dots,P_{j-2},\underline{P}_{j-1},\overline{P}_j,\dots,\overline{P}_n),$$

because by TM, $f(P_1,\dots,P_{j-2},\overline{P}_{j-1},\overline{P}_j,\dots,\overline{P}_n) \geq f(P_1,\dots,P_{j-2},\underline{P}_{j-1},\overline{P}_j,\dots,\overline{P}_n)$ and, by (12), $f(P_1,\dots,P_{j-1},\overline{P}_j,\dots,\overline{P}_n) > \tau(P_{j-1})$. Thus, $f(P_1,\dots,P_{j-2},\underline{P}_{j-1},\overline{P}_j,\dots,\overline{P}_n) > a_{j-1}$; and, repeating the step for all $i = 1,\dots,j-2$, we get that $f(\underline{P}_1,\dots,\underline{P}_{j-1},\overline{P}_j,\dots,\overline{P}_n) > a_{j-1}$, a contradiction. Therefore, $f(\mathbf{P}) = a_{j-1} = m^{2n-1}(\tau(P_1),\dots,\tau(P_n),a_1,\dots,a_{n-1})$.

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