Theoretical Economics 15 (2020), 1307–1333 1555-7561/20201307

Transferable utility and demand functions

Pierre-André Chiappori
Department of Economics, Columbia University

Elisabeth Gugl
Department of Economics, University of Victoria

While many theoretical works, particularly in family economics, rely on the transferable utility (TU) assumption, its exact implications in terms of individual preferences have never been fully worked out. In this paper, we provide a set of necessary and sufficient conditions for a group to satisfy the TU property. We express these conditions in terms of both individual indirect utilities and individual demand functions. Last, we describe the link between this question and a standard problem in consumer theory (initially raised by Gorman 1953), and explain why a similar characterization in terms of direct utilities cannot obtain.

Keywords. Transferable utility, matching, Gorman polar form, characterization of preference profiles, many private and public goods.

JEL classification. C7, D1, D6.

1. Introduction

Many results in microeconomics and game theory require that preferences exhibit the transferable utility (TU) property, whereby, for a well chosen cardinalization of utilities, the Pareto frontier, defined in the utility space by the set of Pareto extremum pairs of individual utilities, is a straight line with slope -1 (or, for more than two agents, a hyperplane orthogonal to the unit vector) for all price and income bundles, and generally for all economic environments. For instance, in collective models of the household (Chiappori 1988, 1992), TU implies that household (aggregate) demand does not depend on Pareto weights; this allows one to reconcile the unitary model with an explicit representation of individual preferences while addressing issues of intrahousehold redistribution (and inequality).[1] Another important property of TU preferences is that Pareto frontiers do not intersect when prices, incomes, or other factors change; this property, in turn, is

Pierre-André Chiappori: pc2167@columbia.edu
Elisabeth Gugl: egugl@uvic.ca
Chiappori gratefully acknowledges financial support from the National Science Foundation (Grant 1124277). We are indebted to seminar participants at Oxford and London, as well as to the three anonymous referees, for useful comments. Errors are ours.

[1] In particular, Becker's well known "rotten kid" theorem (Becker 1991) can be formulated in a TU framework; see, for instance, Bergstrom (1989) and Browning et al. (2014) for a general discussion.

© 2020 The Authors. Licensed under the Creative Commons Attribution-NonCommercial License 4.0. Available at https://econtheory.org. https://doi.org/10.3982/TE3715

crucial for the so-called Coase theorem to hold.[2] Last, a host of recent works on the market for marriage refer to a frictionless matching framework in the style of Becker (1973) and Shapley and Shubik (1971). In this framework, under TU, the notion of stability is equivalent to total surplus maximization, a fact that considerably simplifies theoretical and empirical analysis.

Yet, while the TU assumption is made on a regular basis, its exact meaning is not clear. Consider, for example, a two-person household consuming private and public goods under a budget constraint. What do we need to assume on individual preferences to get the TU property?

Partial answers have been given to this problem.

- When all commodities are privately consumed, then Bergstrom and Varian (1985) show that preferences must be of the Gorman polar form. We discuss this further in Section 3.4.

- Bergstrom and Cornes (1983) analyze a model in which *all commodities but one* are publicly consumed, and show that, for a given price and income vector, the efficient bundle of public goods does not change as we move from one Pareto efficient allocation to another if and only if preferences are of the generalized quasi-linear (GQL) form. One can readily check that GQL preferences imply TU (a property that also holds in the general case with many public and private goods).

- Finally, Bergstrom (1989), still considering the case with only one private good, shows that if the demand for public goods is the same for all efficient allocations, then TU requires GQL preferences.

To the best of our knowledge, however, the fact that, in Bergstrom and Cornes's context with one private good only, TU implies *both* GQL *and* identical demand for public goods for all efficient allocations has not been proved. In the general case, with an arbitrary number of private and public goods, this statement would actually be incorrect; GQL is sufficient for TU but not necessary. Indeed, Gugl (2014) provides an example of utility functions that are not GQL but lead to TU; we give a more general class of such examples that includes Gugl's original one, in Section 5. So far, however, no *general characterization* of utilities leading to TU in the general case has been provided.[3]

The goal of this note is to fill this gap. We provide a set of testable conditions that are implied by the TU assumption in a market context. We first refer to the notion of conditional indirect utility introduced by Blundell et al. (2005), defined as the maximum utility level an individual can reach by choosing the optimal bundle of private consumption for given values of prices of private goods and total private expenditures, and conditional on a given vector of public consumption. We introduce a specific property of individual preferences, the affine conditional indirect utility (ACIU), which states that for a well

[2]A typical application is the Becker–Coase theorem, which states that divorce laws should have no impact on divorce rates; see Chiappori et al. (2015).

[3]Bergstrom (1997) conjectures a result similar to our Theorem 1. However, to the best of our knowledge, no proof has been provided yet. Our paper provides a formal proof of the indirect utility characterization and, moreover, a general characterization in terms of demand functions.

chosen cardinal representation, the conditional indirect utility is affine in total private expenditures, and we show that TU obtains if and only if (i) each individual preference exhibits the ACIU property and (ii) the coefficient of total private expenditures (which can be a function of prices of private goods and consumption of public goods) is the same for all individuals. We show that this result generalizes the previous results in the sense that it boils down to a Gorman polar form in the absence of consumption of public goods and to GQL with only one private good; moreover, we provide an example that is neither Gorman polar form nor GQL, but satisfies our characterization and, therefore, generates TU. Finally, we show that the equivalence between TU and the invariance of demand for public goods over efficient allocations holds only for the one private good case. In general, preferences satisfying the second property need not satisfy TU; we actually provide examples in which almost all do not.

We then analyze how the ACIU property translates in terms of direct utility, and, more importantly, in terms of conditional and unconditional demand functions. Regarding direct utilities, we show that ACIU is related to the standard notion of Gorman polar forms. Specifically, the ACIU property requires that direct utilities, *considered as functions of private commodities alone*, admit a Gorman polar form representation with income coefficients identical across agents (whereas the dependence on consumption of public goods is not constrained). It is well known from standard consumer theory, at least in the case of private goods, that for most preferences admitting a Gorman polar form representation, the corresponding utilities cannot be expressed in closed form; consequently, most direct utilities compatible with TU preferences do not admit a closed-form representation.

Translating the ACIU property in terms of individual and aggregate demand functions is arguably more interesting, since the latter are empirically observable—so that the conditions can actually be tested. We derive the implications of ACIU for both conditional and unconditional individual demands. Moreover, these conditions can be extended to the group's aggregate demand function. While TU implies that groups behave as individuals, the converse is not true: even if a group maximizes a single utility, its aggregate behavior cannot be derived from a TU model if it fails to satisfy the conditions we describe. At any rate, it is now possible to test, from observable behavior, whether individual and/or aggregate preferences satisfy the TU property.

Last, we provide an illustration of our results on specific functional forms for demands. We consider the standard linear expenditures system (LES) and show that the corresponding *individual* demands always satisfy the ACIU property; we actually exhibit a broad generalization of LES systems that also implies ACIU. This suggests that the requirements imposed by TU on the form of individual demands, although restrictive, are not necessarily extreme; in particular, they are much more lax than those required by the GQL form.[4] However, TU also imposes restrictions on the level of heterogeneity between agents. Specifically, we show that for *a set of* individual LES demands to be compatible with TU, many (but not all) coefficients must be equal across individuals.

[4] Chiappori (2010) shows that a LES demand system can *never* be of the GQL form when there are several private goods.

This suggests that the major cost of assuming TU lies in the limitations imposed on the nature and importance of heterogeneity within the group (and their consequences in terms of group behavior).

2. The framework

For expositional clarity, we present our model in the case of two agents and use language that refers to members of a family. However, our results generalize to any number of agents (see Section 3.3) and apply to other groups as well such as subnational governments in the context of fiscal federalism.

A *household*, consisting of the *wife* w and the *husband* h, consume n private and N public commodities that can be purchased on a market.[5] Let $x_m = (x_m^1, \dots, x_m^n)$, where $m = h, w$, denotes member m's private consumption and $p = (p^1, \dots, p^n)$ denotes the corresponding price vector. Similarly, $X = (X^1, \dots, X^N)$ denotes the household's consumption of public goods purchased at price $P = (P^1, \dots, P^N)$. Throughout the note, the vector $\pi = (p^1, \dots, p^n, P^1, \dots, P^N)$ belongs to a compact subset $\mathcal{P}$ of $\mathbb{R}_+^{n+N}$, while the vectors $(x_h^1, \dots, x_h^n)$, $(x_w^1, \dots, x_w^n)$ and $(X^1, \dots, X^N)$ belong to $\mathbb{R}_+^n$ and $\mathbb{R}_+^N$, respectively. In particular, prices and consumptions are assumed strictly positive. Finally, let $y > 0$ denote the household's total income, so that the household's budget constraint is

$$\sum_{i=1}^{n} p^i \left(x_h^i + x_w^i\right) + \sum_{j=1}^{N} P^j X^j = y.$$

We assume away consumption externalities; therefore, m's preferences depend only on the vector (X, x_m). Moreover, we assume that agents' preferences can each be represented by utility functions $u_h(X, x_h)$ and $u_w(X, x_w)$ that are twice continuously differentiable, strictly increasing, and strictly quasi-concave.

We can now provide a precise definition of the notion of transferable utility (TU).

Definition 1. We say that a pair of husband and wife preferences satisfy the transferable utility (TU) property if there exist two strictly increasing, twice continuously differentiable, strictly quasi-concave utility functions $\bar{u}_h$ and $\bar{u}_w$, representing, respectively, husband and wife preferences, such that for any given vector of prices and total income (P, p, y), *all* Pareto efficient allocations (X, x_h, x_w) satisfy the condition

$$\bar{u}_h(X, x_h) + \bar{u}_w(X, x_w) = K(P, p, y)$$

for some function K. If this is the case, then we say that the pair of utilities $(\bar{u}_h, \bar{u}_w)$ satisfies the TU condition.

A key remark is that the TU property is ordinal; preferences satisfy the TU condition if one can find specific cardinal representations, $\bar{u}_h$ and $\bar{u}_w$, such that, for all values of

[5] The goods are public within the household only; they are privately purchased on the market. One may think of housing or expenditures on children as typical examples.

prices and income, the Pareto frontier, defined in the utility space by the set of Pareto extremum pairs of individual utilities, is a straight line with slope equal to -1. Obviously, the same preferences can be represented by other "cardinalizations"; for any such C^2 cardinalization, say, u_h and u_w, the TU property requires that there exist two continuously differentiable, strictly monotonic functions F_h and F_w such that for any given vector of prices and total income (P, p, y), all Pareto efficient allocations (X, x_h, x_w) satisfy the condition

$$F_h \circ u_h(X, x_h) + F_w \circ u_w(X, x_w) = K'(P, p, y)$$

for some function K'. Moreover, note that since

$$\bar{u}_h + \bar{u}_w = K(P, p, y),$$

with transforms such that

$$u_m = G_m(\bar{u}_m), \quad m = h, w,$$

it must be true that

$$G_h^{-1} \circ u_h(X, x_h) + G_w^{-1} \circ u_w(X, x_w) = K(P, p, y).$$

Last, if both u_h and u_w are affine transforms of $\bar{u}_h$ and $\bar{u}_w$, then

$$u_m = \alpha_m \bar{u}_m + \beta_m, \quad m = w, h,$$

with $\alpha_h, \alpha_w > 0$, and, moreover, if $\alpha_h = \alpha_w$, then F_h and F_w can be taken to be the identity mapping. These particular cardinalizations, for which the property is directly satisfied, play a special role. In particular, we systematically use one of these cardinalizations to represent individual preferences.

Next, following Blundell et al. (2005), for any twice continuously differentiable, strictly increasing, and strictly quasi-concave utility u_m, we define the corresponding *conditional indirect utility* v_m by

$$\forall (X, p, \rho_m) \in \mathbb{R}_+^{n+N+1}, \quad v_m(X, p, \rho_m) = \max_{x_m} u_m(X, x_m) \quad \text{such that } p' x_m = \rho_m, \tag{1}$$

where $p' x_m$ denotes the scalar product of vectors p and x_m.[6] Note that under our assumptions, v_m is continuously differentiable.

The economic interpretation of this notion is the following. Assume the household decision process results in a vector X being purchased, while a monetary amount ρ_m is left available for m's expenditure on private goods. Then $v_m(X, p, \rho_m)$, the conditional indirect utility of agent m $(m = w, h)$, represents the maximum utility m can reach by

[6]Note that since $p \in \mathbb{R}_+^n$, the space $\{x_m \mid p' x_m = \rho_m\}$ is compact; since u_m is continuous, the function v_m is well defined. Moreover, since u_m is strictly quasi-concave, the maximum is reached for a unique bundle. As always, both the direct utility and the conditional indirect utility are defined up to a strictly increasing transformation. That is, if the direct utility u_m and the conditional indirect utility v_m represent preferences, then for any continuously differentiable, strictly increasing transformation F_m, the transformed utilities $\tilde{u}_m = F_m \circ u_m$ and $\tilde{v}_m = F_m \circ v_m$ represent the same preferences.

optimally selecting the private consumption vector x_m under the constraint $p'x_m = \rho_m$. The *conditional private demand* vector $\xi_m(X, p, \rho_m)$ is defined as the unique solution to (1): it is the vector of private consumption goods that maximizes m's utility for prices p, private expenditures ρ_m, and consumption of public goods X.

The main motivation for that notion is given by the following result.

Proposition 1. *Let $(\bar{X}, \bar{x}_h, \bar{x}_w)$ be a Pareto efficient allocation, and define $\bar{\rho}_m = p'\bar{x}_m$. Then*

$$u_m(\bar{X}, \bar{x}_m) = v_m(\bar{X}, p, \bar{\rho}_m) \quad \textit{and} \quad \bar{x}_m = \xi_m(X, p, \bar{\rho}_m), \quad m = h, w.$$

Proof. Assuming not, without loss of generality, let w receive $\bar{x}_w \neq \xi_w$. Then by the definition of ξ_w it must be the case that

$$u_w(\bar{X}, \bar{x}_w) < u_w(\bar{X}, \xi_w).$$

But then the allocation $(\bar{X}, \bar{x}_h, \xi_w)$ satisfies the budget constraint and strictly Pareto-dominates $(\bar{X}, \bar{x}_h, \bar{x}_w)$, a contradiction. □

The underlying intuition is that any Pareto efficient decision mechanism can be represented as a two-stage process. In the first stage, agents jointly decide on the level X of consumption of public goods and on the distribution of the remaining resources between agents; i.e., the wife gets ρ_w and the husband gets $\rho_h = y - P'X - \rho_w$ (according to the standard vocabulary of collective models, the pair (ρ_w, ρ_h) defines a *sharing rule*). In the second stage, agents each choose their optimal vector of private consumption, conditional on the vector X and subject to the budget constraint $p'x_m = \rho_m$. The conditional indirect utility reflects the outcome of the second stage, conditionally on the level of public goods and individual private expenditures defined in the first stage. The latter, which characterizes Pareto efficient allocations of household resources between public and private consumptions, can then be written as

$$\max_{X, \rho_w} v_w(X, p, \rho_w)$$

under the constraint

$$v_h\big(X, p, y - P'X - \rho_w\big) \geq \bar{v}_h \tag{2}$$

for some given $\bar{v}_h$.

Then the TU property requires the following attribute: one can find specific cardinalizations (v_w, v_h) of individual preferences such that, for all Pareto efficient allocations, the Lagrange multiplier of the constraint (2) is equal to 1, irrespective of the price–income bundle and of the parameter $\bar{v}_h$.

3. The main result

In this section, we state our main result and corollaries. We show that our results generalize to more than two agents.

3.1 *The statement*

We can now state our main result. It is based on the following definition.

Definition 2. A utility function u_m satisfies the affine conditional indirect utility (ACIU) property if one can find a twice continuously differentiable scalar function $\alpha_m(X, p)$ from $\mathbb{R}^{N+n}$ to $\mathbb{R}$ that is (-1)-homogeneous in p, and a twice continuously differentiable scalar function $\beta_m(X, p)$ from $\mathbb{R}^{N+n}$ to $\mathbb{R}$ that is 0-homogeneous in p, such that the conditional indirect utility corresponding to u_m can be written as

$$v_m(X, p, \rho) = \alpha_m(X, p)\rho_m + \beta_m(X, p) \quad \text{for all } (X, p, \rho_m). \tag{3}$$

In other words, the corresponding, conditional indirect utility is an affine function of the variable ρ_m.[7]

The result is now the following theorem.

Theorem 1. *A pair of husband and wife utilities satisfy the TU property if and only if they both satisfy the ACIU property:*

$$\forall (X, p, \rho_m), \quad v_m(X, p, \rho) = \alpha_m(X, p)\rho_m + \beta_m(X, p), \quad m = h, w$$

with moreover

$$\alpha_h(X, p) = \alpha_w(X, p) \quad \textit{for all } (X, p). \tag{4}$$

Proof. We first show that (3) and (4) are sufficient. Assume they are satisfied, and consider any Pareto efficient allocation $(\bar{X}, \bar{x}_h, \bar{x}_w)$ such that $0 < \bar{\rho}_w = p'\bar{x}_w < y - P'\bar{X}$. Then the allocation must be such that $\bar{\rho}_w = p'\bar{x}_w$ solves a program of the form

$$\max_{\rho_w} \alpha(\bar{X}, p)\big((1-\mu)\rho_w + \mu\big(y - P'\bar{X}\big)\big) + \beta_w(\bar{X}, p) + \mu\beta_h(\bar{X}, p), \tag{5}$$

where μ denotes the Lagrange multiplier of the constraint. For any interior solution, this requires $\mu = 1$, implying that

$$u_w + u_h = \max_X \big[\alpha(X, p)\big(y - P'X\big) + \beta_w(X, p) + \beta_h(X, p)\big].$$

Conversely, assume that the TU property holds and consider the program (where p is omitted for brevity)

$$\max_{X, \rho_w} v_w(X, \rho_w) + v_h\big(X, y - P'X - \rho_w\big). \tag{6}$$

For notational convenience, we present the argument for the case of a single public good ($N = 1$); the extension to $N \geq 2$ is straightforward. The crucial remark is that *the solution*

[7]The homogeneity properties of α_m and β_m follow from the general properties of the indirect utility function, which is homogenous of degree 0 in (p, ρ_m).

to this program cannot be unique (since all Pareto efficient allocations solve it). It follows that the first order conditions

$$\frac{\partial v_w}{\partial \rho_w}(X, \rho_w) - \frac{\partial v_h}{\partial \rho_h}(X, y - PX - \rho_w) = 0,$$
$$\frac{\partial v_w}{\partial X}(X, \rho_w) + \frac{\partial v_h}{\partial X}(X, y - PX - \rho_w) - P\frac{\partial v_h}{\partial \rho_h}(X, y - PX - \rho_w) = 0,$$

considered as two equations in (X, ρ_w), have a continuum of solutions (the set of solutions typically is a one-dimensional manifold). This implies either that one equation (in that case, the first) is degenerate or that the two equations are redundant (technically, they are not *transversal*). We consider both cases successively.

Assume, first, that

$$\frac{\partial v_w}{\partial \rho_w}(X, \rho_w) - \frac{\partial v_h}{\partial \rho_h}(X, y - PX - \rho_w) = 0$$

for all values of (X, ρ_w, P, y). Then consider the change in variables

$$(X, \rho_w, P, y) \to (X, \rho_w, P, t) \quad \text{where } t = y - PX.$$

Then

$$\frac{\partial v_w}{\partial \rho_w}(X, \rho_w) - \frac{\partial v_h}{\partial \rho_h}(X, t - \rho_w) = 0 \tag{7}$$

for all t. Differentiating in t yields that $\partial v_h/\partial \rho_h$ is independent of ρ_h, so is $\partial v_w/\partial \rho_w$ by (7), which proves the result.

Assume now that the equations are redundant, and define

$$F(X, \rho_w, P, t) = \frac{\partial v_w}{\partial \rho_w}(X, \rho_w) - \frac{\partial v_h}{\partial \rho_h}(X, t - \rho_w),$$
$$G(X, \rho_w, P, t) = \frac{\partial v_w}{\partial X}(X, \rho_w) + \frac{\partial v_h}{\partial X}(X, t - \rho_w) - P\frac{\partial v_h}{\partial \rho_h}(X, t - \rho_w)$$

with the same change in variables as before. Since $\partial F/\partial P = 0$, it must be the case that $\partial G/\partial P = 0$; therefore, $\partial v_h/\partial \rho_h = 0$, which is a particular case of the previous case. □

In words, Theorem 1 states that for TU to obtain, a pair of husband and wife utilities must have the following properties. First, fix the vector X of consumption of public goods and consider individual utilities as functions of private consumptions only. Then the corresponding, individual (conditional) indirect utilities must be (possibly after a twice continuously differentiable, strictly increasing transformation) affine in total private expenditures for all values of prices and income. Second, the coefficient of total private expenditures (which may depend on prices of private goods, but also on consumption of public goods) must be the same for all agents. These conditions are necessary and sufficient: utilities that satisfy these conditions also satisfy TU, and any utilities that satisfy TU must satisfy these properties.

The result can, equivalently, be expressed in terms of (conditional) expenditures functions $e_m(X, p, u_m)$, defined as the minimum amount that agent m, facing prices p and endowed with a vector X of consumption of public goods, must spend on private goods to reach a utility level u_m. Then the following corollary holds.

Corollary 1. *A pair of husband and wife utilities satisfy the TU property if and only if for each member $m = h, w$, there exist two differentiable, 1-homogeneous scalar functions $\alpha'_m(X, p)$ and $\beta'_m(X, p)$ from $\mathbb{R}^{N+n}$ to $\mathbb{R}$ such that the conditional expenditure function takes the form*

$$e_m(X, p, u_m) = \alpha'_m(X, p)u_m + \beta'_m(X, p)$$

for all m, with moreover $\alpha'_h(X, p) = \alpha'_w(X, p)$.

Proof. The proof directly stems from the fact that $e_m(X, p, u)$ is the inverse (in ρ_m) of $v_m(X, p, \rho_m)$; therefore,

$$v_m(X, p, \rho_m) = \alpha_m(X, p)\rho_m + \beta_m(X, p) \quad \Longleftrightarrow \quad e_m(X, p, u) = \frac{u}{\alpha_m(X, p)} - \frac{\beta_m(X, p)}{\alpha_m(X, p)}.$$

The homogeneity properties of α'_m and β'_m follow immediately. □

An important corollary of Theorem 1 and Proposition 1 is the following.

Corollary 2. *If preferences satisfy the TU property, then, for any (P, p, y), the household's demand for public goods is the same for all Pareto efficient allocations.*

The proof of Corollary 2 follows directly from Theorem 1 and Proposition 1, since for any efficient allocation, X solves

$$\max_X \big[\alpha(X, p)\big(y - P'X\big) + \beta_h(X, p) + \beta_w(X, p)\big].$$

It should be noted, however, that our results are needed to establish this result. Indeed, under TU, any Pareto efficient allocation must solve

$$\max_{X, \rho_w} v_w(X, p, \rho_w) + v_h\big(X, p, y - P'X - \rho_w\big).$$

TU implies that this maximization program has a continuum of solutions (all efficient allocations, typically a one-dimensional manifold). However, what is not clear is whether they all correspond to the same demand for public goods; in principle, it could be the case that different efficient allocations correspond to different values of the sharing rule ρ *and* different demands for public goods X (one for each possible choice of ρ). The corollary states that such a situation is not possible.

3.2 *Two counterexamples*

Note that the converse of Corollary 2 is false: one can find preferences such that (i) all *interior* Pareto efficient demands for public goods are identical,[8] while (ii) these preferences *do not* satisfy the TU property. We provide two counterexamples.

3.2.1 *Different Cobb–Douglas preferences* Our first counterexample is the following. Assume $n = 2$, $N = 1$, and individual direct utilities are Cobb–Douglas,

$$u_m(X, x_m^1, x_m^2) = X^{\theta_m} x_m^1 (x_m^2)^{\delta_m}, \quad m = h, w,$$

where we assume that $\delta_h \neq \delta_w$ and $\theta_h \neq \theta_w$, but

$$\frac{\theta_w}{1+\delta_w} = \frac{\theta_h}{1+\delta_h} = \lambda.$$

Prices of the private goods are $p^1 = 1$ and $p^2 = p$, and P denotes the price of the public good.

Proposition 2. *Given the preferences specified above, the following statements hold.*

(i) The demand for the public good is the same for all Pareto efficient allocations. It is given by

$$X = \frac{\lambda}{1+\lambda}\frac{y}{P}.$$

(ii) For the appropriate cardinalization, the absolute value of the slope of the Pareto frontier is constant and is given by

$$\frac{dv_w}{dv_h} = p^{\frac{\delta_h - \delta_w}{(1+\delta_w)(1+\delta_h)}}.$$

For the proof, see the Appendix.

Thus for the appropriate cardinalization, the Pareto frontier is a straight line, but its slope is not constant: it depends on the price of the second good, which invalidates TU.

3.2.2 *Exclusive goods* Another example that generates a demand for public goods that is the same for all (interior) Pareto efficient allocations is the case of identical Cobb–Douglas preferences, where one of the private goods is leisure, given by $x_m^n \in [0, 1]$. Let the wage rate of member m be given by ω_m. Let the exponent on the public good be λ, let the exponent on leisure be δ, let the exponent for private good i be θ^i, and assume that all the exponents of private goods including leisure add up to 1, such that

$$\delta = 1 - \sum_{i=1}^{n-1} \theta^i.$$

[8]One can easily exhibit examples in which arbitrary preferences generate identical demands for public goods only because a corner solution is reached in all cases. We are obviously not interested in these situations.

The utility function is given by

$$u_m(X, x_m) = X^{\lambda}\left(x_m^n\right)^{\delta} \prod_{i=1}^{n-1} x_m^i{}^{\theta^i}, \quad m = h, w.$$

For any ρ_h, ρ_w, and nonwage income $\overline{y}$,

$$\rho_h + \rho_w = \overline{y} + \omega_h + \omega_w - PX.$$

PROPOSITION 3. *Given the preferences specified above, the following statements hold.*

(i) The demand for the public good is the same for all Pareto efficient allocations. It is given by

$$X = \frac{\lambda}{1+\lambda}\frac{y}{P}.$$

(ii) The absolute value of the slope of the Pareto frontier is constant and given by

$$\frac{dv_w}{dv_h} = \left(\frac{\omega_h}{\omega_w}\right)^{\delta}.$$

See the Appendix for the proof.

Thus the Pareto frontier is a straight line, but its slope is not constant: it depends on the agents' wages, which invalidates TU.

3.2.3 *Interpretation* The intuition underlying both examples can be grasped by referring to the result by Bergstrom and Cornes (1983). Consider, for instance, the first example, and assume u_h and u_w generate a demand for public goods that is the same for all Pareto efficient allocations; let $v_h(X, p, \rho_h)$ and $v_w(X, p, \rho_w)$ denote the corresponding conditional indirect utilities. Fix the price vector to some arbitrary value $\bar{p}$ and define $u_m(X, \rho_m) = v_m(X, \bar{p}, \rho_m)$, $m = h, w$. Then $u_m(X, \rho_m)$ can be considered as a direct utility in an economy with N public goods X and one private good ρ. Since (u_h, u_w) are such that the demand for public goods is the same for all Pareto efficient allocations, by Bergstrom and Cornes (1983) these utilities must thus have a GQL cardinalization; therefore, v_m must be of the form

$$v_m(X, \bar{p}, \rho_m) = F_m\left[a(X, \bar{p})\rho_m + b_m(X, \bar{p}), \bar{p}\right] \tag{8}$$

for some smooth mappings a, b_m, and F_m, where F_m is strictly increasing in its first argument. In particular, using the adequate normalization, the Pareto frontier is linear. Note, however, that, in general, the cardinalizations, and therefore the slope of the Pareto frontier, depend on $\bar{p}$ (technically, $\bar{p}$ is, in general, an argument of the functions F_m), whereas *TU requires the slope to be price- (or wage-) independent.*

In other words, if for any given $\bar{p}$ one can find a ($\bar{p}$-dependent) cardinalization that gives the ACIU property, then the demand for public goods is the same for all efficient

allocations. The key point, however, is that TU cannot obtain unless *the same cardinalization works for all* $\bar{p}$, which is not the case in general. In the first example, the set of parameters for which the identical demand property obtains is

$$S = \left\{ \delta_h, \theta_h, \delta_w, \theta_w \text{ s.t. } \frac{\theta_h}{1+\delta_h} = \frac{\theta_w}{1+\delta_w} \right\}.$$

The set of parameters for which the TU property obtains, alternatively, is

$$S' = \{\delta_h, \theta_h, \delta_w, \theta_w \text{ s.t. } \delta_h = \delta_w \text{ and } \theta_h = \theta_w\}.$$

Note that $S' \subset S$ and that, moreover, the measure of S' within S is zero. Also, the same argument holds for the second example, replacing $\bar{p}$ with wages $(\bar{w}_h, \bar{w}_w)$. In fact, our second example shows that with exclusive goods, *none* of the Cobb–Douglas preferences satisfying the uniform public demand property is compatible with TU.

This result illustrates how specific the one private good case is. Given (8), by homogeneity, the price of the unique private good can always be normalized to 1; then the cardinalization is automatically price-independent, and one gets an equivalence between TU and identical demand for public goods across efficient allocations. But this conclusion holds only in that case. Even in our two simple examples, either *almost all* (in the first case) or *all* (in the second case) preferences generating identical demand for public goods are non-TU.

3.3 *Extension: The case of S agents*

The previous result can readily be extended to any number of agents. Consider a group of S agents who consume n private and N public commodities that can be purchased on a market.[9] Let $x_m = (x_m^1, \dots, x_m^n)$, where $m = 1, \dots, S$ denotes member m's private consumption and $p = (p^1, \dots, p^n)$ denotes the corresponding price vector. Similarly, $X = (X^1, \dots, X^N)$ denotes the group's consumption of public goods purchased at price $P = (P^1, \dots, P^N)$.

Definition 3. A set of S individual preference relationships satisfies the transferable utility (TU) property if there exists S strictly increasing, twice continuously differentiable, strictly quasi-concave utility functions $u_1, \dots, u_S$, where u_s represents preferences of agent s, such that for any given vector of prices and total income (P, p, y), *all* Pareto efficient allocations $(X, x_1, \dots, x_S)$ satisfy the condition

$$\sum_{s=1}^{S} u_s(X, x_s) = K(P, p, y)$$

for some function K. If this is the case, then we say that the set of utilities $(u_1, \dots, u_S)$ satisfies the TU condition.

The main result is given by the following theorem.

[9]The goods are public within the group only; they are privately purchased on the market.

THEOREM 2. *A set of individual utilities satisfies the TU property if and only if they all satisfy the ACIU property*

$$\forall (X, p, \rho_s), \quad v_s(X, p, \rho_s) = \alpha_s(X, p)\rho_s + \beta_s(X, p), \quad s = 1, \dots, S$$

with moreover

$$\alpha_s(X, p) = \alpha_t(X, p) \quad \forall s, t \in \{1, \dots, S\}.$$

PROOF. For sufficiency, the S agents analogue to (5) is

$$\max_{\rho_1,\dots,\rho_{S-2},\rho_S} \left[\begin{array}{c} \alpha(\bar{X}, p)\left((1-\mu_{S-1})\rho_S + \mu_{S-1}\big(y - P'\bar{X}\big) + \sum_{t=1}^{S-2}(\mu_t - \mu_{S-1})\rho_t\right) \\ +\beta_S(\bar{X}, p) + \sum_{t=1}^{S-1}\mu_t\beta_t(\bar{X}, p) \end{array} \right],$$

and for necessity, the S agents analogue to (6) is

$$\max_{X,\rho_1,\dots,\rho_{S-1}} \sum_{s=1}^{S-1} v_s(X, \rho_s) + v_S\left(X, y - P'X - \sum_{s=1}^{S-1}\rho_s\right).$$

The rest of the proof follows the same logic as in the case of two agents. □

3.4 *Links with existing results*

One can easily check that the form characterized by Theorems 1 and 2 encompasses several results already available in the literature.

- In the absence of public goods, then (3) in Definition 2 boils down to the Gorman polar form (GPF) discussed by Bergstrom and Varian (1985). Indeed, in that case, the conditional indirect utility coincides with the standard indirect utility, since the sharing rule coincides with income:

$$\rho_h = y_h, \rho_w = y_w, \quad \text{s.t. } y_h + y_w = y.$$

- Therefore, m's indirect utility and demand for commodity i are

$$v_m(X, p, y_m) = \alpha(p)y_m + \beta_m(p),$$
$$x_m^i(p, y_m) = -\frac{\partial\alpha/\partial p^i}{\alpha(p)}y_m - \frac{\partial\beta_m/\partial p^i}{\alpha(p)}.$$

  Note, however, that Bergstrom and Varian's result does not extend directly to the case of public goods. The problem here is that applying Bergstrom and Varian's result requires ruling out the possibility of different efficient allocations involving different demands for public goods.

- If, alternatively, there is only one private good ($n = 1$), then we can normalize p to be 1, and we get the generalized quasi-linear (GQL) form of Bergstrom and Cornes (1983); indeed, the sharing rule is then the individual consumption of the private good, and m's *direct* utility becomes

$$u_m(X, x_m) = \alpha_m(X)x_m + \beta_m(X).$$

  Specifically, Bergstrom and Cornes show the following result: when $n = 1$, a necessary and sufficient condition for all Pareto efficient allocations to generate the same demand for public goods is that preferences are GQL, and it is well known that GQL preferences imply TU. Our result also establishes the converse property, namely that when $n = 1$, TU requires GQL preferences, implying that X must be identical across all efficient allocations.

- The GQL form can readily be extended to $n \geq 2$ by considering direct utility functions of the form

$$u_m\big(X, x_m^1, \dots, x_m^n\big) = \alpha_m(X)x_m^1 + \beta_m\big(X, x_m^2, \dots, x_m^n\big).$$

  One can readily check that the corresponding, conditional indirect utility is affine in the sharing rule (this property comes directly from the fact that for a given vector of public goods, u_m as a function of $x_m^1, \dots, x_m^n$ is quasi-linear). For $\alpha_h = \alpha_w$, it therefore implies TU. The converse, however, is not true; indeed, Section 5 provides an example of preferences that are not GQL but still imply TU.

4. Direct utilities and demands

The ACIU property is expressed in terms of *indirect* (conditional) utilities. Indirect utilities are extremely convenient in applied consumer theory, if only because they allow a direct derivation of demand functions (through Roy's identity); conversely, it is, in general, easy to recover the functional form of indirect utilities from that of demand functions. This contrasts starkly with direct utilities. For many direct utilities, demand functions cannot be derived in closed form; conversely, most demand functions used in empirical microeconomics (starting with the most standard ones, i.e., Deaton and Muellbauer's (1980) almost ideal demand system (AIDS) and its quadratic extension (QUAIDS) do not admit a general, closed-form representation for direct utilities whereas they admit a closed-form representation for indirect utilities).

Still, it is interesting to consider what the ACIU property implies for the shape of direct utilities. As it happens, most direct utilities corresponding to ACIU preferences do not have a closed-form representation (although their existence can be established by standard arguments); moreover, there exists a close relationship between this problem and a standard question in consumer theory, namely the characterization of preferences that admit a Gorman polar form representation. In the next subsection, we explore this relationship and show in particular how our results generalize Gorman's initial insights.

A second, more empirically relevant question relates to the implications of ACIU for the shape of *demand functions*. Unlike direct or indirect utilities, demand functions

are empirically observable; therefore, a characterization of their form has a potentially direct impact on econometric approaches to problems involving transferable utility. In the second subsection, we first provide a characterization of conditional demands and then derive an equivalent formulation for unconditional demand systems. Throughout this section, the individual index m is omitted for brevity.

4.1 *Direct utilities*

We first start with the following definition.

Definition 4. Given some direct utility $u(X, x)$, the X-conditional direct utility is the mapping u^X defined by

$$x \to u^X(x) = u(X, x).$$

Given the assumptions made on u, u^X is twice continuously differentiable, strictly increasing, and strongly quasi-concave. Note also that u^X is the direct utility corresponding to the conditional indirect utility v,

$$\forall (X, p, \rho) \in \mathbb{R}_+^{n+N+1}, \quad v(X, p, \rho) = \max_x u^X(x) \quad \text{s.t. } p'x = \rho,$$

or, conversely,

$$u^X(x) = \min_{p,\rho} v(X, p, \rho) \quad \text{s.t. } p'x = \rho.$$

Last, the conditional demand function $\xi(X, p, \rho)$ solves

$$\max_x u^X(x) \quad \text{s.t. } p'x = \rho. \tag{9}$$

It is well known that under the assumptions we have made, ξ is continuously differentiable almost everywhere.

Now the previous results imply that a direct utility function $u(X, x)$ is compatible with ACIU if and only if, for all X, u^X admits a Gorman polar form representation with income coefficients identical across agents. That is, there must exist some smooth scalar functions α and β such that

$$u(X, x) = \min_p \big[\alpha(X, p)p'x + \beta(X, p)\big].$$

It is well known that, unlike indirect utilities, *direct* utilities admitting a Gorman polar form representation do not have an explicit closed form in general (see, for instance, Blackorby et al. 1978). In the next section, however, we provide an example for which a closed form exists.

4.2 *Demand functions*

From an empirical perspective, the ACIU property, which characterizes indirect utilities, cannot be directly tested, since the latter are not directly observable. It is therefore

useful to characterize its implications for demand functions, which can (in principle) be empirically recovered. This can be done in two different ways: one can derive the conditional or unconditional demands.

4.2.1 *Conditional individual demands for private goods* One may first consider conditional demands for private goods, $\xi(X, p, \rho)$, defined earlier as the optimal bundle of private consumption for an individual faced with prices p, and endowed with a vector X of public goods and a monetary amount ρ to be spent on private consumption. Note that we have an equivalent to Roy's identity:

$$\xi^i(X, p, \rho) = -\frac{\partial v(X, p, \rho)/\partial p^i}{\partial v(X, p, \rho)/\partial \rho}. \tag{10}$$

In practice, conditional demands can be estimated from demand data by regressing demand on prices, demands for public goods, and total expenditures on private goods. The latter variables are obviously endogenous, and must, therefore, be instrumented; a standard solution is to use as instruments total income and the prices of public goods. The following corollary provides a simple restriction on the functional forms that can be used.

Corollary 3. *If a utility function satisfies the ACIU property, then there exist two C^2 scalar functions α, which is (-1)-homogeneous in p, and β, which is 0-homogeneous in p, such that the conditional private demand vector $\xi(X, p, \rho)$ is of the form*

$$\xi^i(X, p, \rho) = a^i(X, p)\rho + b^i(X, p), \tag{11}$$

where

$$a^i(X, p) = -\frac{\partial \alpha(X, p)/\partial p^i}{\alpha(X, p)}, \qquad b^i(X, p) = -\frac{\partial \beta(X, p)/\partial p^i}{\alpha(X, p)}.$$

Conversely, if there exist two C^2 scalar functions α, which is (-1)-homogeneous in p, and β, which is 0-homogeneous in p, such that the conditional private demand vector $\xi(X, p, \rho)$ is of the form (11), then the corresponding preferences satisfy the (ACIU) property.

Proof. As noted in (10), the envelope theorem applied to (1) gives a conditional version of Roy's identity. Plugging (3) into this relationship gives the result. Conversely, under the assumptions made, the function v defined by

$$v(X, p, \rho) = \alpha(X, p)\rho + \beta(X, p), \quad \forall (X, p, \rho)$$

is such that, for $i = 1, \dots, n$,

$$\xi^i(X, p, \rho) = -\frac{\partial v(X, p, \rho)/\partial p^i}{\partial v(X, p, \rho)/\partial \rho}. \qquad \square$$

In other words, conditional individual demands for private goods must be affine in total expenditures on private goods.

4.2.2 *Unconditional individual demands for private goods* Alternatively to $\xi(X, p, \rho)$, we may consider standard, "unconditional" Marshallian demands $x(P, p, y)$, $X(P, p, y)$, which are functions of (all) prices and household income. It is well known that under standard conditions, these functions are continuously differentiable almost everywhere.

Recall that total expenditures on private goods is given by ρ:

$$\rho(P, p, y) = \sum_i p^i x^i(P, p, y).$$

Define matrix $J^{X,\rho}$ by

$$J^{X,\rho} = \begin{pmatrix} \frac{\partial X^1}{\partial P^1} & \cdots & \frac{\partial X^1}{\partial P^N} & \frac{\partial X^1}{\partial y} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial X^N}{\partial P^1} & \cdots & \frac{\partial X^N}{\partial P^N} & \frac{\partial X^N}{\partial y} \\ \frac{\partial \rho}{\partial P^1} & \cdots & \frac{\partial \rho}{\partial P^N} & \frac{\partial \rho}{\partial y} \end{pmatrix}.$$

In other words, $J^{X,\rho}$ is the Jacobian matrix of the vector $(X^1, \dots, X^N, \rho)$ with respect to variables $(P^1, \dots, P^N, y)$. A point $\bar{\pi} = (\bar{P}, \bar{p}, \bar{y})$ is said to be *regular* if $J^{X,\rho}$ is invertible at $\bar{\pi}$; then $J^{X,\rho}$ is invertible on some open neighborhood $\mathcal{O}(\bar{\pi})$ of $\bar{\pi}$. By the implicit function theorem, we can then (locally) express $(P^1, \dots, P^N, y)$ as a function of $(X^1, \dots, X^N, \rho)$; the Jacobian matrix of this function is simply the inverse $(J^{X,\rho})^{-1}$. Let d denote the last column of $(J^{X,\rho})^{-1}$; that is, d is the vector of derivatives of $(P^1, \dots, P^N, y)$ with respect to ρ.

The relationship between Marshallian and conditional demands for good i is

$$\xi^i(X, p, \rho) = x^i(P, p, y) \quad \forall (P, p, y),$$

which implies that

$$\frac{\partial \xi^i}{\partial \rho} = \sum_k \frac{\partial x^i}{\partial P^k} \frac{\partial P^k}{\partial \rho} + \frac{\partial x^i}{\partial y} \frac{\partial y}{\partial \rho}.$$

In matrix terms,

$$D_\rho \xi = J^x d,$$

where

$$J^x = \begin{pmatrix} \frac{\partial x^1}{\partial P^1} & \cdots & \frac{\partial x^1}{\partial P^N} & \frac{\partial x^1}{\partial y} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial x^n}{\partial P^1} & \cdots & \frac{\partial x^n}{\partial P^N} & \frac{\partial x^n}{\partial y} \end{pmatrix}$$

is the Jacobian matrix of the vector of *private* consumptions $(x^1, \dots, x^n)$ with respect to $(P^1, \dots, P^N, y)$.

Then we have the following result.

Proposition 4. *Assume that some individual utility satisfies the ACIU property. Then, in a small enough neighborhood of any regular point $\bar{\pi}$, the vector $J^x d$ can be expressed as a function of (X, p) only; equivalently, the Jacobian matrix of $J^x d$ with respect to (P, y) must be orthogonal to the vector d.*

In the last section, we give a linear expenditures system (LES) example; we show that LES demands are compatible with TU if and only if some of the coefficients are identical across agents.

4.2.3 *Unconditional household demands for private goods* A consequence of the previous results is that, under TU, different Pareto efficient allocations may correspond to different vectors of individual private consumptions for all private commodities. This contrasts with a standard property of GQL demands (for which individual demands for all private goods except one are identical over the set of Pareto efficient allocations).

However, the *household* conditional demand for private good i is

$$\begin{aligned}\xi^i(X, p, \rho) &= a(X, p)\Big(\sum_{m=h,w} \rho_m\Big) + \sum_{m=h,w} b_m(X, p)\\ &= a(X, p)\big(y - P'X\big) + \sum_{m=h,w} b_m(X, p),\end{aligned}$$

which is identical over all Pareto efficient allocations. A first implication is that, as is well known, TU requires groups to behave as single individuals; in particular, the group's demand must satisfy income pooling and Slutsky symmetry and negativeness.

In addition, the conditions on demand functions that we just derived at the individual level are stable by aggregation. Formally, we now consider a household demand function $(x(P, p, y), X(P, p, y))$ in the neighborhood of some regular point $\bar{\pi} = (\overline{P}, \overline{p}, \bar{y})$. We define $\rho = p'x(P, p, y)$ as the household's aggregate expenditures on private goods, define $J^{X,\rho}$ as the matrix of derivatives of the vector $(X^1, \dots, X^N, \rho)$ with respect to variables $(P^1, \dots, P^N, y)$, define J^x as the matrix of derivatives of the vector of *household* private consumptions $(x^1, \dots, x^n)$ with respect to $(P^1, \dots, P^N, y)$, and define d as the last column of $(J^{X,\rho})^{-1}$. Then we have the following result.

Proposition 5. *Assume that a pair of husband and wife utilities satisfy the TU property. Then there exists a differentiable, strictly increasing, strictly quasi-concave utility function such that the group's aggregate demand maximizes this utility under the group's budget constraint. In particular, the group's aggregate demand satisfies Slutsky symmetry and semi-negativeness. Moreover, the household's Marshallian demand $x(P, p, y)$ for private commodities is such that the vector $J^x d$ can be expressed as a function of (X, p) only; that is, the Jacobian matrix of $J^x d$ with respect to $(P^1, \dots, P^N, y)$ is orthogonal to the vector d.*

In particular, we see that while TU implies that the group behaves as a single decision maker (i.e., the group's aggregate behavior can be derived from the maximization of a unique utility function), the converse is absolutely not true: aggregate demand functions stemming from the maximization of a unique utility are *not* compatible with TU if they fail to satisfy the properties described in Theorem 1.

4.3 *Econometric tests*

How can these properties be empirically tested? Start with the characterization of conditional individual demands. Assume one has consumption data involving public and private goods. The basic idea is to estimate a demand system for private goods using the consumed public goods as conditioning variables (and taking total expenditures on private goods as the income variable). Clearly, endogeneity is an issue in that context, because any exogenous shock (affecting either preferences or any other unobserved factor) is likely to affect the demand for private and for public goods, as well as total expenditures. We therefore need a set of instruments. As is standard in applied microeconomics works, total expenditures can be instrumented by household income. Regarding the consumption of public goods, there exist obvious candidates, namely, the prices of the various public goods. Indeed, these prices obviously affect the demand for public goods, but the conditional demand function $\xi(X, p, \rho)$ solves program (9) above, which, *conditional on X and ρ*, does not depend on prices of the public goods.

The empirical strategy could thus rely on the following steps:

- Estimate the conditional demand function $\xi(X, p, \rho)$ using prices of all public goods and total income as instrument for X and ρ.
- Test whether the conditional demand functions are linear or affine in ρ.
- If so, test whether the coefficient of ρ is the same across individuals.

It should be stressed that this approach requires that data provide sufficient variations in the prices of public goods. Note that standard theory implicitly assumes they do, since it considers the matrix of demand derivatives with respect to prices. Clearly, this leaves open an interesting but very difficult question, namely, Can the TU property be tested in the absence of price variations for public goods? We already know that even in the simpler context of individual demand, it is impossible to test for utility maximization, let alone recovering individual demands, when prices do not vary (or vary in a perfectly correlated manner). Whether it is nevertheless possible to test some complex properties (such as the ones described in the paper) remains an open question.

Finally, one can alternatively test the properties on the Marshallian demand, using the predictions of Proposition 5. A possible difficulty is that one needs to compute the inverse of the Jacobian matrix $J^{X,\rho}$, which may not admit a closed-form representation (it does not in the case of the almost ideal demand system). Then it has to be numerically computed.

4.3.1 *Links with Chiappori* Last, our findings generalize the results of Chiappori (2010), who considers exclusively GQL forms. This can be seen at different levels. Consider, first, Chiappori's "conditional quasilinearity" property, which states that the X-conditional utility must be quasi-linear. If this property is satisfied, then the corresponding conditional indirect utility will be affine in the sharing rule, so that our conditions are satisfied. But clearly they are more general: quasi-linear preferences obviously generate Gorman polar form (conditional) demands, but most Gorman polar form (conditional) demands come from preferences that are not quasi-linear. Similarly, it is easy to check

that the conditions in Proposition 3 of Chiappori (2010) imply the conditions of our Proposition 5, while the converse is not true.

Last, one can find examples of direct utilities that are not GQL, but satisfy our conditions. We provide an explicit example in the next section.

5. Example

Throughout this section, the individual index m is omitted for brevity.

5.1 *An example of a linear expenditure system*

As a first illustration, consider the utility function

$$u(X, x) = \prod_i \left(x^i - \gamma^i\right)^{c^i} \prod_j \left(X^j - \Gamma^j\right)^{C^j},$$

which generates LES demands

$$p^i x^i = \gamma^i p^i + c^i (y - \gamma - \Gamma),$$
$$P^j X^j = \Gamma^j P^j + C^j (y - \gamma - \Gamma),$$

where $\gamma = \sum_i \gamma^i p^i$, $\Gamma = \sum_j \Gamma^j P^j$, and $\sum_i c^i + \sum_j C^j = 1$. Defining $c = \sum_i c^i$ and $C = \sum_j C^j$ (then $c + C = 1$), the interpretation of such demands is as follows: there is a minimum amount of each good (for each private good i, this is given by γ^i; for each public good j, this is given by Γ^j) that must be consumed. This leaves a person with $(y - \gamma - \Gamma)$ of disposable income beyond these minimum purchases. A fraction c^i of $(y - \gamma - \Gamma)$ is then spent on a private good i beyond the minimum expenditure required to purchase the minimum amount γ^i and a fraction C^j of $(y - \gamma - \Gamma)$ is then spent on a public good j beyond the minimum expenditure required to purchase the minimum amount Γ^j. Note that if there are no minimum amounts, the LES demands become the demands of a Cobb–Douglas utility function.

The LES specification leading to TU is useful when family economists consider intertemporal choice. Given more than one period, naturally more than one private good enters agents' utility functions. In intertemporal choice models, family members are often assumed to consume one private good and one public good in each period. If the instantaneous utility function is quasi-linear where private consumption enters the utility function linearly, then the model exhibits a constant marginal rate of substitution between the private consumption in different periods—an unrealistic assumption. Alternatively, these instantaneous utility functions give rise to TU in each period and intertemporally (see Gugl and Welling 2017). Hence TU provides a straightforward benchmark for efficiency as efficiency is independent of distribution. The LES specification provides an attractive alternative because it captures a decreasing marginal rate of substitution between private goods in different periods while still yielding a Pareto efficient allocation that is independent of distribution intertemporally as well as in any given period.

Let us now derive the conditional demands. We have

$$\rho = \sum_i p^i x^i = \gamma + c(y - \gamma - \Gamma);$$

therefore,

$$\frac{\rho - \gamma}{c} = y - \gamma - \Gamma$$

and

$$p^i x^i = \gamma^i p^i + \frac{c^i}{c}(\rho - \gamma).$$

The corresponding, conditional indirect utility is, therefore,

$$\begin{aligned} v(X, p, \rho) &= \prod_i \left(\frac{c^i}{c} \frac{\rho - \gamma}{p^i} \right)^{c^i} \prod_j (X^j - \Gamma^j)^{C^j} \\ &= \left(\frac{\rho - \gamma}{c} \right)^c \prod_i \left(\frac{c^i}{p^i} \right)^{c^i} \prod_j (X^j - \Gamma^j)^{C^j}. \end{aligned}$$

We see that

$$[v(X, p, \rho)]^{\frac{1}{c}} = \frac{\rho - \sum_i \gamma^i p^i}{\sum_i c^i} \prod_i \left(\frac{c^i}{p^i} \right)^{\frac{c^i}{c}} \prod_j (X^j - \Gamma^j)^{\frac{C^j}{c}},$$

which is affine in ρ. The coefficient of ρ is

$$K = \prod_i \left(\frac{c^i}{p^i} \right)^{\frac{c^i}{c}} \prod_j (X^j - \Gamma^j)^{\frac{C^j}{c}} / c$$

and it is the same for all agents if and only if the c^i, C^j, and Γ^j are the same for all agents. Note, however, that the parameters γ^i, corresponding to the minimum consumptions for private goods, may be individual specific.

We conclude that any LES demand system is compatible with TU, provided that preferences are similar enough across agents. A very interesting remark is that, by a result owing to Chiappori (2010), a LES demand cannot possibly be of the GQL form unless there is only one private good. This further illustrates the fact that the set of preferences compatible with TU is much larger than the set of GQL preferences. It also shows that TU requires assumptions regarding the form of individual demands that, although strong, are not particularly extreme and may be satisfied by some usual functional forms. In the end, the really restrictive assumptions are related to the level of heterogeneity between agents: TU does impose strong constraints on how much preferences can vary across agents.

5.2 *Generalizing LES*

As is well known, LES preferences, at least when all goods are private, are among the few preferences that admit a Gorman polar form representation and a closed-form utility function. Specifically, if we assume, in the previous example, that $N = 0$ (so that all goods are private), then the corresponding utility is

$$u(x) = \prod_i \left(x^i - \gamma^i\right)^{c^i}.$$

We now provide an example of utility functions for private and public goods that generalizes both the LES system and the example provided by Gugl (2014). The corresponding preferences are neither Gorman polar form nor GQL; but they still satisfy ACIU, therefore, TU, and still admit a closed-form representation for the direct utility. We partition the n private goods into a group of n_1 private goods indexed from 1 to n_1 and another group indexed $n_1 + 1, \dots, n$.

Consider the utility

$$u(X, x) = \frac{a(X)}{\Psi} \prod_{k=1}^{n_1} \left(x^k - \gamma^k(X)\right)^{c^k} + b\left(X, x^{n_1+1}, \dots, x^n\right) \tag{12}$$

with

$$\sum_{k=1}^{n_1} c^k = 1 \quad \text{and} \quad \Psi = \prod_{k=1}^{n_1} \left(c^k\right)^{c^k}.$$

The conditional indirect utility of (12) is defined by

$$v(X, p, \rho) = \max_x \frac{a(X)}{\Psi} \prod_{k=1}^{n_1} \left(x^k - \gamma^k(X)\right)^{c^k} + \beta\left(X, x^{n_1+1}, \dots, x^n\right) \tag{13}$$

under

$$\sum_{i=1}^{n} p^i x^i = \rho.$$

The maximand in (13) is separable in $(x^1, \dots, x^{n_1})$; therefore, the program can be solved using two-stage budgeting. Define

$$\varrho = \sum_{k=1}^{n_1} p^k x^k. \tag{14}$$

Then $(x^1, \dots, x^{n_1})$ must solve

$$\max_{(x^1,\dots,x^{n_1})} \prod_{k=1}^{n_1} \left(x^k - \gamma^k(X)\right)^{c^k}$$

under (14), which gives

$$x^k = \gamma^k(X) + \frac{c^k}{p^k}\left(\varrho - \sum_{i=1}^{n_1} p^i \gamma^i(X)\right).$$

The first stage is, therefore,

$$\max_{\varrho, x^{n_1+1}, \dots, x^n} \left(a(X) \prod_{k=1}^{n_1} \left(p^k\right)^{-c^k}\right)\left(\varrho - \sum_{i=1}^{n_1} p^i \gamma^i(X)\right) + b\left(X, x^{n_1+1}, \dots, x^n\right)$$

under

$$\varrho + \sum_{k=n_1+1}^{n} p^k x^k = \rho.$$

The maximand is quasi-linear in ϱ. Assuming that $\varrho > 0$, this implies that $(x^{n_1+1}, \dots, x^n)$ depends only on (p, X), so that

$$b\left(X, x^{n_1+1}, \dots, x^n\right) = B(X, p),$$

$$\varrho = \rho - \sum_{k=n_1+1}^{n} p^k x^k = \rho - C(X, p).$$

Finally, for individual m, the conditional indirect utility is given by

$$v_m(X, p, \rho_m) = a_m(X) \prod_{k=1}^{n_1} \left(p^k\right)^{-c_m^k} \rho_m + B_m(X, p) - a(X) \prod_{k=1}^{n_1} \left(p^k\right)^{-c^k} \left(C_m(X, p) + \sum_{k=1}^{n_1} p^i \gamma_m^i(X)\right),$$

which has the ACIU property for all prices if and only if, for $k = 1, \dots, n_1$, the c_m^k coefficients are identical across agents and $a_h(X) = a_w(X) = a(X)$. The corresponding Pareto frontier is, therefore, defined by

$$u_h + u_w = K(P, p, y),$$

where

$$K(P, p, y) = \max_X \left[\begin{array}{c} a(X) \prod_{k=1}^{n_1} \left(p^k\right)^{-c^k} \left(y - P'X\right) \\ + \sum_{=h,w} \left(B_m(X, p) - a(X) \prod_{k=1}^{n_1} \left(p^k\right)^{-c^k} \left(C_m(X, p) + \sum_{k=1}^{n_1} p^i \gamma_m^i(X)\right)\right) \end{array}\right].$$

6. Conclusion

Household economics has experienced spectacular changes over the recent decades. On the one hand, the development of the collective approach has lead to an explicit "formalization" of power relationships within the household. Econometric tools now exist for the empirical estimation of models that recognize the specific (public vs. private) nature of various goods and services. Several such models have been estimated (see, for instance, Cherchye et al. 2012, Lise and Yamada 2019, or Chiappori et al. 2020, among many others). Moreover, important advances have been made regarding both the theoretical representation and the econometric estimation of economies of scale in consumption and intrahousehold inequality (see, for instance, Browning et al. 2013 or Dunbar et al. 2013). On the other hand, the analysis of marital patterns has attracted renewed attention; most of the corresponding models (such as Chiappori et al. 2018, Ciscato et al. 2020, or Chiappori et al. 2019 to name just a few) use a transferable utility framework in which individual demands for several commodities are explicitly modeled and estimated. The question studied in the present paper lies at the intersection of these two branches of literature. Ultimately, one would like to develop the joint estimation of household formation (or dissolution) and behavior from a matching model involving a general consumption structure. Such a goal requires, as a first step, a complete characterization of individual demand models that are compatible with the TU assumption. As such, the present paper is a step in a promising research direction.

Appendix

A.1 *Proof of Proposition 2*

Given the utility function in the example of Section 3.2.1, the conditional indirect utilities are

$$v_m(X, p, \rho) = X^{\theta_m} K_m \rho_m^{1+\delta_m} p^{-\delta_m},$$

where $K_m = \frac{\delta_m^{\delta_m}}{(1+\delta_m)^{1+\delta_m}}$. An alternative and more convenient cardinalization is

$$\bar{v}_m(X, p, \rho_m) = \left(\frac{v_m(X, p, \rho_m)}{K_m}\right)^{\frac{1}{1+\theta_m}} = X^{\frac{\theta_m}{1+\delta_m}} \rho_m p^{-\frac{\delta_m}{1+\delta_m}} = X^{\lambda} \rho_m p^{-\frac{\delta_m}{1+\delta_m}}.$$

In particular, while this function is affine (indeed, linear) in ρ_m, the coefficient of ρ_m is different for w and h unless $\delta_h = \delta_w$. Therefore, if $\delta_h \neq \delta_w$, these preferences do *not* satisfy the TU property.

Yet, an efficient level of public good solves

$$\begin{aligned}
&\max_{X,\rho_w} v_w(X, p, \rho_w) + \mu v_h(X, p, y - PX - \rho_w) \\
&\quad = \max_{X,\rho_w} X^{\lambda} \rho_w p^{-\frac{\delta_w}{1+\delta_w}} + \mu X^{\lambda} (y - PX - \rho_w) p^{-\frac{\delta_h}{1+\delta_h}}.
\end{aligned}$$

An interior solution cannot obtain unless

$$p^{-\frac{\delta_w}{1+\delta_w}} = \mu p^{-\frac{\delta_h}{1+\delta_h}}, \quad \text{i.e. } \mu = p^{\frac{\delta_h-\delta_w}{(1+\delta_w)(1+\delta_h)}},$$

which proves part (ii) of Proposition 2.

The program then boils down to

$$\max_X X^\lambda (y - PX) \quad \text{which yields } X = \frac{\lambda}{1+\lambda}\frac{y}{P},$$

which proves part (i) of Proposition 2; the demand for the public good is the same for all (interior) Pareto efficient allocations.

A.2 *Proof of Proposition 3*

Given the utility function in the example in Section 3.2.2, the conditional indirect utilities are

$$v_m(X, \omega_m, \rho_m) = X^\lambda K \rho_m \omega_m^{-\delta},$$

where

$$K = \delta^\delta \prod_{i=1}^{n-1} \left(\frac{\theta^i}{p^i}\right)^{\theta^i}.$$

An efficient level of public good solves

$$\begin{aligned}\max_{X,\rho_w} & v_w(X, p, \rho_w) + \mu v_h(X, p, y - PX - \rho_w) \\ &= \max_{X,\rho_w} X^\lambda K \rho_w \omega_w^{-\delta} + \mu X^\lambda K \rho_h \omega_h^{-\delta}.\end{aligned}$$

An interior solution cannot obtain unless

$$\omega_w^{-\delta} = \mu \omega_h^{-\delta}, \quad \text{i.e., } \mu = \left(\frac{\omega_h}{\omega_w}\right)^\delta.$$

Hence,

$$\frac{dv_w}{dv_h} = \left(\frac{\omega_h}{\omega_w}\right)^\delta,$$

which proves part (ii) of Proposition 3.

The program then boils down to

$$\max_X X^\lambda (y - PX), \quad \text{which yields } X = \frac{\lambda}{1+\lambda}\frac{y}{P},$$

which proves part (i) of Proposition 3; the demand for the public good is the same for all (interior) Pareto efficient allocations.

References

Becker, Gary S. (1973), "A theory of marriage: Part 1." *Journal of Political Economy*, 81, 813–846. [1308]

Becker, Gary S. (1991), *A Treatise on the Family*. Harvard University Press, Cambridge, Massachusetts. [1307]

Bergstrom, Theodore C. (1989), "A fresh look at the Rotten Kid theorem—and other household mysteries." *Journal of Political Economy*, 97, 1138–1159. [1307, 1308]

Bergstrom, Theodore C. (1997), "A survey of theories of the family." In *Handbook of Population and Family Economics* (Mark R. Rosenzweig and Oded Stark, eds.), 21–79, North Holland, Amsterdam, The Netherlands. Volume 1, Part A. [1308]

Bergstrom, Theodore C. and Richard Cornes (1983), "Independence of allocative efficiency from distribution in the theory of public goods." *Econometrica*, 51, 1753–1765. [1308, 1317, 1320]

Bergstrom, Theodore C. and Hal Varian (1985), "When do market games have transferable utility?" *Journal of Economic Theory*, 35, 222–233. [1308, 1319]

Blackorby, Charles, Richard Boyce, and R. Robert Russell (1978), "Estimation of demand systems generated by the gorman polar form; A generalization of the S-branch utility tree." *Econometrica*, 46, 345–363. [1321]

Blundell, Richard, Pierre-André Chiappori, and Costas Meghir (2005), "Collective labor supply with children." *Journal of Political Economy*, 113, 1277–1306. [1308, 1311]

Browning, M., Chiappori P.A., and Weiss Y. (2014), *Economics of the Family*. Cambridge University Press, Cambridge, UK. [1307]

Browning, Martin, Pierre-André Chiappori, and Arthur Lewbel (2013), "Estimating consumption economies of scale, adult equivalence scales, and household bargaining power." *Review of Economic Studies*, 80, 1267–1303. [1330]

Cherchye, Laurens, Bram De Rock, and Frederic Vermeulen (2012), "Married with children: A collective labor supply model with detailed time use and intrahousehold expenditure information." *American Economic Review*, 102, 3377–3405. [1330]

Chiappori, Pierre-André (1988), "Rational household labor supply." *Econometrica*, 56, 63–90. [1307]

Chiappori, Pierre-André (1992), "Collective labor supply and welfare." *Journal of Political Economy*, 100, 437–467. [1307]

Chiappori, Pierre-André (2010), "Testable implications of transferable utility." *Journal of Economic Theory*, 145, 1302–1317. [1309, 1325, 1326, 1327]

Chiappori, Pierre-André, Monica Costa Dias, and Costas Meghir (2018), "The marriage market, labor supply, and education choice." *Journal of Political Economy*, 126, S26–S72. [1330]

Chiappori, Pierre-André, Monica Costa Dias, Costas Meghir, and Pengpeng Xiao (2019), "Education, marriage and child development." Working Paper, Yale University. [1330]

Chiappori, Pierre-André, Murat Iyigun, and Yoram Weiss (2015), "The Becker–Coase theorem reconsidered." *Journal of Demographic Economics*, 81, 157–177. [1308]

Chiappori, Pierre-André, Costas Meghir, and Yoko Okuyama (2020), "Intrahousehold welfare: Theory and empirical application." Unpublished paper, Department of Economics, Columbia University. [1330]

Ciscato, Edoardo, Alfred Galichon, and Marion Goussé (2020), "Like attract like? A structural comparison of homogamy across same-sex and different-sex households." *Journal of Political Economy*, 128, 740–781. [1330]

Deaton, Angus and Muellbauer John (1980), *Economics and Consumer Behavior*. Cambridge University Press, Cambridge. [1320]

Dunbar, Geoffrey R., Arthur Lewbel, and Krishna Pendakur (2013), "Children's resources in collective households: Identification, estimation and an application to child poverty in Malawi." *American Economic Review*, 103, 438–471. [1330]

Gugl, Elisabeth (2014), "Transferable utility in the case of many private and many public goods." *Studies of Microeconomics*, 2, 133–140. [1308, 1328]

Gugl, Elisiabeth and Linda Welling (2017), "Efficiency of family bargaining models with renegotiation: The role of transferable utility across periods." *Studies in Microeconomics*, 5, 53–83. [1326]

Lise, Jeremy and Ken Yamada (2019), "Household sharing and commitment: Evidence from panel data on individual expenditures and time use." *The Review of Economic Studies*, 86, 2184–2219. [1330]

Shapley, Lloyd S. and Martin Shubik (1971), "The assignment game I: The core." *International Journal of Game Theory*, 1, 111–130. [1308]

Co-editor Dilip Mookherjee handled this manuscript.

Manuscript received 7 April, 2019; final version accepted 21 January, 2020; available online 30 January, 2020.