

Markov distributional equilibrium dynamics in games with complementarities and no aggregate risk

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We present a new approach to studying equilibrium dynamics in a class of stochastic games with a continuum of players with private types and strategic complementarities. We introduce a suitable equilibrium concept, called *Markov Stationary Nash Distributional Equilibrium* (MSNDE), prove its existence, and determine comparative statics of equilibrium paths and the steady-state invariant distributions to which they converge. Finally, we provide numerous applications of our results including: dynamic models of growth with status concerns, social distance, and paternalistic bequests with endogenous preferences for consumption.

KEYWORDS. Large games, distributional equilibria, supermodular games, comparative dynamics, non-aggregative games, social interactions.

JEL CLASSIFICATION. C62, C72, C73.

1. INTRODUCTION

This paper presents a constructive method for characterizing and approximating Markovian equilibria in a class of dynamic games with a continuum of players and

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strategic complementarities. Each player is endowed with a private type that evolves stochastically over time. The type may be interpreted as the agent's endowment, social rank, payoff-relevant private information, behavioral traits, etc., depending on the economic application at hand. An equilibrium is defined as a probability distribution over types and actions of all players in the initial period, and a law of motion/belief regarding future distributions of types and actions in the population. Our approach allows for a unified study of both equilibrium transition paths and equilibrium comparative dynamics from any initial state of the game, as well as associated long-run stochastic steady states (i.e., invariant distributions) to which these equilibrium paths converge.

Large dynamic games with private information find numerous applications in diverse fields in economics, including models of equilibrium growth with heterogeneous agents and endogenous social structure (Cole, Mailath, and Postlewaite (1992)), inequality with endogenous preference formation (Genicot and Ray (2017)), industry dynamics with heterogeneous firms (Weintraub, Benkard, and Van Roy (2008)), dynamic network formation (Mele (2017), Xu (2018)), economics of identity and social dissonance (Akerlof and Kranton (2000), Bisin, Moro, and Topa (2011)), models of endogenous formation of social norms (Acemoglu and Jackson (2017)), macroeconomic models with public or private sunspots (Angeletos and Lian (2016)), or models of wealth distribution in the presence of incomplete markets (Cao (2020)).¹ The principal objective of these papers is to provide sufficient conditions for existence, characterization, computation, calibration, and estimation of dynamic equilibria. However, the existing tools are typically applicable only to the study of *stochastic steady-state* equilibria defined in terms of invariant distributions. In contrast, we provide a systematic method for studying *global* equilibrium distribution transitional paths that converge to the stochastic steady states.

The theoretical literature on equilibrium dynamics in games is quite limited, even in the context of games with finitely many players.² This is because characterizing the dynamics of sequential or Markovian equilibria becomes analytically intractable as the number of players grows and the state space becomes large and complex. Additionally, due to heterogeneity of private types, determining how players update their beliefs both “on” and “off” equilibrium paths is nontrivial. Even providing sufficient conditions for existence of sequential equilibria is challenging, let alone providing methods for approximating and characterizing the evolution of types and actions over time.

Due to these complications, the literature has focused on equilibrium concepts that simplify dynamic interactions. Two dominant methodological approaches have been proposed in the existing literature. One exploits *aggregative structures* in games and restricts players' interactions to a statistic or an aggregate that summarizes the population distribution, joint with some notion of a stochastic steady-state equilibrium. The other (often used in conjunction with aggregation and restriction to steady states) simplifies

¹See also Acemoglu and Jensen (2015, 2018) for a discussion on the relation between large dynamic economies and large anonymous games.

²From a theoretical perspective, little is known about the nature of convergence of equilibrium transitional dynamics to stochastic steady states. This question is complicated by the presence of equilibrium multiplicities and stability issues related to equilibrium transitional paths, thus making the counterfactuals from these models difficult to implement and interpret.

equilibrium interactions by imposing behavioral assumptions on how agents make their decisions. This includes the recent work on *oblivious equilibria* (Lasry and Lions (2007), Achdou, Buera, Lasry, Lions, and Moll (2014), Bertucci, Lasry, and Lions (2019), Light and Weintraub (2021), Achdou, Han, Lasry, Lions, and Moll (2022)), *mean-field equilibria* (Weintraub, Benkard, and Van Roy (2008), Adlakha, Johari, and Weintraub (2015), and Ifrach and Weintraub (2016)), or *imagined-continuum equilibria* (Kalai and Shmaya (2018)), among others.³ We argue that such simplifications are not critical or necessary in games with strategic complementarities.

Our results This paper tackles the above theoretical and computational questions within a unified methodological framework of large anonymous stochastic games with strategic complementarities⁴ and no aggregate risk.⁵ Exploiting the nature of games with infinitely many agents, where individuals have negligible impact on actions of others, and in the context of games with strategic complementarities, we provide sufficient conditions for existence of a *Markov stationary Nash distributional equilibrium* (henceforth MSNDE). Our solution concept consists of a probability measure over types and actions in the population of players and a law of motion that determines the equilibrium evolution of such distributions. MSNDE is defined over a *minimal* set of state variables⁶ and resembles the notion of recursive competitive equilibrium that is extensively used in macroeconomics. Our equilibrium concept is therefore inherently dynamic and enables us to characterize and compare equilibrium transition paths. Notably, the results hold without the need of restricting our analysis to an aggregative structure. In fact, in our economic applications, players' payoffs critically depend on the *entire* distribution of types and actions in the population.⁷

Limiting our attention to games with strategic complementarities is indispensable for our results. First of all, it allows us to formulate the evolution of (distributional) equilibrium beliefs in a tractable way. Second, we develop a new order-theoretic approach to characterize the order structure of (Markovian) Nash distributional equilibria. This enables us to prove existence of a greatest and a least MSNDE (with respect to a well-defined stochastic order). Third, by analyzing a measure space of agents, we avoid the technical difficulties that can emerge in extensive-form supermodular games with a finite number of players and private information.⁸ Finally, our approach delivers novel

³See also Krusell and Smith (1998) where agents know only the moments of the random measure determining the distribution of idiosyncratic shocks and assets. See also Lacker (2020), Doncel, Gast, and Gaujal (2016), Nutz (2018), Kwok and Ho (2019).

⁴See Topkis (1978), Vives (1990), Milgrom and Roberts (1990) and Milgrom and Shannon (1994) for early contributions and motivations for studying games with strategic complementarities. See also Van Zandt and Vives (2007) and Van Zandt (2010) for Bayesian games with strategic complementarities and finitely many players.

⁵See Jovanovic and Rosenthal (1988), Bergin and Bernhardt (1992), Karatzas, Shubik, and Sudderth (1994).

⁶By the *minimal state space*, we mean a domain that includes only the current individual type and the distribution of types in the population.

⁷Equilibrium distributions are also important in econometric evaluations of heterogeneous agent models in macroeconomics. See, for example, Parra-Alvarez, Posch, and Wang (2020) and Auclert, Bardoczy, Rognlie, and Straub (2021).

⁸See Echenique (2004), Vives (2009), and Mensch (2020) for a related discussion.

results in constructing *global* equilibrium comparative statics/dynamics that apply to monotone dynamic economies. This way, we extend the recent results on comparative statics of stochastic steady-state equilibria as in Acemoglu and Jensen (2015) and Light and Weintraub (2021).

We organize the paper as follows. In the remainder of this section, we present a motivating example to discuss the main issues that we tackle in our results. In Section 2, we present the main model and prove equilibrium existence. The results on monotone comparative statics/dynamics are discussed in Section 3. We present economic applications of our results in Section 4. Proofs, auxiliary results, and a glossary of mathematical terminology are postponed until the Appendix.

A motivating example Consider a growth model in which individuals are concerned with their relative social status. The society consists of a continuum of players. Each time period $n \in \{1, 2, \dots\}$, a typical player is endowed with a (private) wealth/capital level $t \in T = [0, 1]$ that constitutes their type. This wealth can be transformed into consumption $c \in [0, 1]$ or investment $a \in A = [0, 1]$ using a one-to-one technology, thus introducing the constraint $t = c + a$. By investing $a \in [0, t]$, the agent influences their wealth t' in the following period via a stochastic technology q . Whenever a units of wealth is being invested, the cumulative probability of attaining the capital t' is $q(t'|a)$. We assume that higher investments make higher wealth more likely, that is, the distribution $q(\cdot|a)$ increases in a in the sense of first-order stochastic dominance. Finally, we assume that the realization of the future capital t' is independent across players. The discount factor, common to all players, is denoted by $\beta \in (0, 1)$.

The status of each agent is determined by both their current consumption c and wealth t . In each period, every individual interacts randomly with one other member of the society. If an agent with capital t consuming c encounters an individual of wealth $\tilde{t}$ consuming $\tilde{c}$, the former receives $U(c, \tilde{c}, t, \tilde{t}) = m(t - \tilde{t}) + w(c - \tilde{c})$, where the functions m and w are continuous, strictly increasing, and concave. Thus interacting with individuals with lower wealth and consumption is preferable due to, say, the feeling of superiority.

At the beginning of each period, before any interaction with other members of society takes place, the individual determines their consumption c and investment a . In order to do so, they first evaluate their beliefs about the current distribution μ over capital-investments pairs $(\tilde{t}, \tilde{a})$ in the population, where $\tilde{a} = \tilde{t} - \tilde{c}$. Given a belief μ , their expected payoff in this particular period is given by

$$r(t, a, \mu) = \int_{A \times T} [m(t - \tilde{t}) + w(t - a - \tilde{t} + \tilde{a})] \mu(d\tilde{a} \times d\tilde{t}).$$

Of particular importance is to notice that the payoffs of each player depends on the *entire distribution* of type-actions in the population. Thus, this game is inherently *nonaggregative*. Indeed, evaluating payoffs in each period requires the entire distribution μ of capital (types) and investments (actions) in the population. Simply replacing the measure μ with a summary statistic is not enough to define the payoffs.

We are interested in studying the dynamic *distributional* equilibrium of this game. More generally, we investigate how the distributions of types and actions in the population evolve and interact when (a) the evolution of types and actions is determined by strategies of individuals and the stochastic transition q over private types, and (b) individuals form beliefs over future types and actions in the population consistent with the law of motion governing the distribution of private types (i.e., capital levels), given the joint strategy of all players.

Studying equilibria in games with infinitesimal players has one important advantage. Since individual players have a negligible impact on the distribution of types and actions in the population, each one of them faces a standard *Markov decision problem* (henceforth, MDP), conditional on the distributions of future types $\{\tau_n\}$ and types-actions $\{\mu_n\}$. Only private types are drawn (independently) each period and fluctuate according to probability distribution q .

Importantly, the problem of each player admits a *recursive formulation*. To see that, suppose the players share a *macro belief* Φ , that is, a transition function for capital-investment distributions between periods, where $\mu_{n+1} = \Phi(\mu_n)$. Together with an initial distribution μ_1 , this allows players to conjecture a candidate equilibrium path of the game and formulate their sequential problem recursively, with the value function v^* satisfying

$$v^*(t, \mu; \Phi) = \max_{a \in [0, t]} \left\{ (1 - \beta)r(t, a, \mu) + \beta \int_T v^*(t', \Phi(\mu); \Phi) q(dt' | a) \right\}.$$

An MSNDE consists of a measure μ^* over type-actions in the population in the initial period and a macrobelief transition Φ^* such that, when treating those as given, almost every player solves their MDP and the resulting distribution over types and actions coincides with μ^* . In particular, the marginal distribution of μ^* over types (wealth/capital) must be equal to the exogenously given initial distribution τ_1 . Moreover, the perceived macro belief Φ^* , under rational expectations, is consistent with the actual transition q and the initial distribution. In particular, under the exact law of large numbers, one can associate probabilities q with distributions over types in T .

In addition, any equilibrium pair (μ^*, Φ^*) generates a sequence of equilibrium measures $\{\mu_n^*\}$, where $\mu_1^* = \mu^*$ and $\mu_{n+1}^* = \Phi^*(\mu_n^*)$. MSNDE is *stationary* in the sense that the corresponding equilibrium strategies of players and their beliefs are independent of time. However, our concept is inherently dynamic and allows us to evaluate and compare the *entire equilibrium paths* of types and actions.

Our motivating example is a game with *dynamic* strategic complementarities. In this game, it is optimal for every individual to increase their own wealth and consumption as the distribution of wealth and consumption in the population increases stochastically. More importantly, such complementarities are present *within* and *across* periods. In particular, anticipating (stochastically) higher distributions of capital tomorrow provides incentives for players to increase their own investment today at the expense of the current consumption. Whether a game exhibits such complementarities depends critically on two reinforcing conditions: (i) increasing differences between the private type (capital) and anticipated population distribution in the following period, and (ii) agents

forming monotone beliefs, that is, expecting higher population distribution tomorrow when faced with a higher distribution today.⁹ This example has both features. This is in contrast to the complementarities that arise in *stochastic steady-state* or *stationary equilibria*, which are essentially static by definition.

Finally, we can characterize the comparative statics/dynamics of equilibrium paths. We show how changes in parameters of the game (e.g., discount factor, preference or technology parameters, the initial distribution τ_1 of types) affect paths of equilibrium distributions $\{\mu_n^*\}$ (implied by equilibrium μ^* and Φ^*), as well as the steady states to which they converge. Since the measure μ_n^* is defined over the space of types *and* actions, it is crucial to provide a novel equilibrium comparative statics result for spaces of *multidimensional* distributions, that extends the existing comparative statics results in a nontrivial fashion.

2. LARGE STOCHASTIC GAMES WITH COMPLEMENTARITIES

In this section, we formally define the model and our notion of equilibrium. A glossary of basic mathematical definitions is provided in the [Appendix](#).

Consider a stochastic game in discrete time with an infinite horizon. Let $(\Lambda, \mathcal{L}, \lambda)$ be a probability space of players. It is critical to our analysis that this space is *superatomless*. This formalization is necessary to show that the agents can form their beliefs about types of other players by exploiting the exact law of large numbers.¹⁰ To fix ideas, one intuitive example of such a space is the product measure space over $[0, 1]^I$, where each factor is endowed with Lebesgue measure and I is uncountable.

In each period $n \in \{1, 2, \dots\}$, a player is endowed with a private type $t \in T \subseteq \mathbb{R}^p$, where T is compact and $\mathcal{T}$ denotes its Borel σ -algebra. Let $A \subseteq \mathbb{R}^k$ be a compact space of all conceivable actions endowed with the Borel σ -algebra $\mathcal{A}$. We endow T and A with the natural product partial order $\geq$.¹¹ Finally, let $\mathcal{M}$ be a set of probability measures on $\mathcal{T} \otimes \mathcal{A}$, and $\mathcal{M}_T$ be the set of probability measures on $\mathcal{T}$, where both spaces are ranked with the corresponding first order stochastic dominance order and endowed with the topology of weak convergence of measures.¹²

Given a distribution τ of types of all (other) players, a player of type t chooses an action $a \in \tilde{A}(t, \tau) \subseteq A$, where $\tilde{A}: T \times \mathcal{M}_T \rightrightarrows A$ is the feasible action correspondence. The player's within-period payoff is determined by a bounded function $r: T \times A \times \mathcal{M} \rightarrow \mathbb{R}$ that takes values $r(t, a, \mu)$, given a private type t , an action a , and a probability measure μ over types and actions of all players.

In this paper, we investigate dynamic games in which private types of players are determined stochastically in each period. The transition probability is represented by a function $q: T \times A \times \mathcal{M} \rightarrow \mathcal{M}_T$ that assigns a probability measure $q(\cdot | t, a, \mu)$ over private types in the following period to their current type t , action a , and the measure μ of types-actions in the population.

⁹Hence, our work is related to recent work on characterizing single crossing in distributions (e.g., [Quah and Strulovici \(2012\)](#) and [Kartik, Lee, and Rappoport \(2019\)](#)).

¹⁰See the [Appendix](#) for a formal definition and a comprehensive discussion on this notion.

¹¹For any $x, y \in \mathbb{R}^\ell$, we say that $x \geq y$ if $x_i \geq y_i$, for all $i = 1, \dots, \ell$.

¹²See Remark A.1 in the [Appendix](#) for a discussion regarding the relationship between order topology and weak convergence of measures.

2.1 Players' decision problem

In order to define the sequential decision problem for each player, we need to specify how the individual is forming beliefs about future types of other players in the game, based on the current distribution of types and strategies in the population. We begin with some basic assumptions on the primitives of the game.¹³

ASSUMPTION 1. For all $\tau \in \mathcal{M}_T$ and $\mu \in \mathcal{M}$:

- (i) The correspondence $t \Rightarrow \tilde{A}(t, \tau)$ is measurable and compact-valued.
- (ii) The function $(a, t) \rightarrow q(\cdot|t, a, \mu)$ is Borel-measurable.

Given that the measure space of players is superatomless, Assumption 1 guarantees that the (endogenous) transition of private types satisfies a *no aggregate uncertainty* condition in each period (hence, it evolves deterministically).¹⁴ Formally, given the current distribution μ of types and actions in the population, the future measure of players with private types in a measurable set S is

$$\phi(\mu)(S) := \int_{T \times A} q(S|t, a, \mu) \mu(dt \times da). \quad (1)$$

Therefore, it is determined by the current distribution μ and the transition q . This *no aggregate uncertainty* condition for a superatomless probability space of players follows from results in Sun (2006) and Podczeck (2010). Such an appropriate *exact law of large numbers* (ELLN) simplifies our analysis of large random populations by (i) allowing for independent draws of types for a continuum of players; (ii) simplifying the dynamics of the aggregate law of motion of distributions over types-actions in the population; and (iii) allowing each agent to form beliefs using the law of large numbers, rather than updating their beliefs on (the product of) types of other players. By ELLN and universality of rich Fubini extension as in Sun (2006), we show the no-aggregate-uncertainty assumption holds in our setting. See Section A.1.2 in the Appendix for details.

Given the formulation of beliefs, we define the decision problem of a player in a candidate Markov stationary Nash distributional equilibrium. Let H_∞ be the set of all histories $\{(t_n, a_n, \tau_n)\}_{n \in \mathbb{N}}$, where $a_n \in \tilde{A}(t_n, \tau_n)$. Let H_n be the set of histories up to time n : that is, $H_n := \{(t_j, a_j, \tau_j)_{j=1}^n : a_j \in \tilde{A}(t_j, \tau_j)\}$. A strategy is a sequence of functions $(\sigma_n)_{n \in \mathbb{N}}$ such that $\sigma_n : H_{n-1} \times T \times \mathcal{M}_T \rightarrow A$ is Borel-measurable in $(t_1, t_2, \dots, t_n) \in T^n$ and $\sigma_n(h_{n-1}, t_n, \tau_n) \in \tilde{A}(t_n, \tau_n)$, where $H_0 = \emptyset$ and the initial values of t_1, τ_1 are given. A strategy $(\sigma_n)_{n \in \mathbb{N}}$ is *Markov* if in each period n , we have $\sigma_n : T \times \mathcal{M}_T \rightarrow A$, that is, the action depends only on the current type $t \in T$ and the current distribution of types $\tau \in \mathcal{M}_T$. Hence, it is history-independent. A Markov strategy is *stationary* if it is time-invariant, that is, we have $\sigma_n = \sigma_{n'}$, for any time periods n, n' .

¹³A glossary containing basic mathematical terminology, including the details regarding the dynamic ELLN in the immediate sequel, is provided in the Appendix.

¹⁴Bergin and Bernhardt (1992, 1995) discuss the importance of this construction.

Given an initial private type t , a public distribution of types τ , and a Markov strategy σ' of other players, a Markov strategy σ induces the unique private measure $P_{t,\tau}^{\sigma,\sigma'}$ on histories of the game. The sequential objective of a player is

$$R(t, (\sigma, \sigma'), \tau) := (1 - \beta) \mathbb{E}_{t,\tau}^{\sigma,\sigma'} \left[r(t, \sigma_1(t, \tau), \mu_1^{\sigma'}) + \sum_{n=2}^{\infty} \beta^{n-1} r(t_n, \sigma_n(t_n, \tau_n), \mu_n^{\sigma'}) \right], \quad (2)$$

where $\beta \in (0, 1)$ is a discount factor, $\mathbb{E}_{t,\tau}^{\sigma,\sigma'}$ is the expectation induced by $P_{t,\tau}^{\sigma,\sigma'}$, and $\mu_n^{\sigma'} = \tau_n(\text{id}_T, \sigma'_n(\cdot, \tau_n))^{-1}$,¹⁵ where id_T denotes the identity function over T . We impose the following assumptions.¹⁶

ASSUMPTION 2 (Payoffs). *The function r is (i) continuous in (t, a) , (ii) monotone sup- and inf-preserving in μ , (iii) increasing in t , (iv) supermodular in a ; and (v) has increasing differences in $(a, (t, \mu))$ and (t, μ) .*

ASSUMPTION 3 (Transition probability). *The transition kernel $q(\cdot|t, a, \mu)$ is (i) continuous in (t, a) , (ii) monotone sup- and inf-preserving in μ , (iii) stochastically increasing in (t, a, μ) , (iv) stochastically supermodular in a , and (v) has stochastically increasing differences in $(a, (t, \mu))$ and (t, μ) .*

ASSUMPTION 4 (Feasible actions). *The correspondence $\tilde{A} : T \times \mathcal{M}_T \rightrightarrows A$ is (i) continuous; (ii) its values are compact sublattices of A ; (iii) it is increasing with t in the sense of set inclusion; (iv) increasing with t in the strong set order; and (v) it satisfies strict complementarities.*

Most of these assumptions are standard in dynamic games with complementarities (see Curtat (1996), Nowak (2007) and Balbus, Reffett, and Woźny (2014)) with the exception of some monotonicity requirements on payoffs and the transition function. As shown later in the paper, these are indispensable to preserve strategic complementarities across periods in the extensive formulation of the game under Markovian strategies. Importantly, our framework encompasses many of the important linear social interaction models studied in the econometric literature by Blume, Brock, Durlauf, and Jayaraman (2015), Kline and Tamer (2020) and Kwok and Ho (2019).¹⁷

An example of a transition function q satisfying Assumption 3 is

$$q(\cdot|t, a, \mu) := g(t, a, \mu)\rho(\cdot) + (1 - g(t, a, \mu))\nu(\cdot),$$

¹⁵We employ the standard notation where, for any measure ν , measurable function f , and a measurable set S , we have $\nu f^{-1}(S) = \nu(\{s : f(s) \in S\})$.

¹⁶We present the mathematical terminology in the Appendix.

¹⁷For example, the payoff function studied in Kwok and Ho (2019) satisfies our assumptions (see equation (1) in their paper). Our constructive monotone comparative statics/dynamics results presented in the following section may be useful in developing and characterizing estimators to test equilibrium distributions in empirical models. See, for example, Echenique and Komunjer (2009, 2013), DePaula (2013), and Uetake and Watanabe (2013).

where $g(t, a, \mu)$ is supermodular in a ; has increasing differences in $(a, (t, \mu))$ and (t, μ) ; and is increasing in (a, t, μ) ; while ρ, ν are probability distributions over $\mathcal{T}$ such that ρ first order stochastically dominates ν . This class of transitions was introduced in [Curtat \(1996\)](#) and [Amir \(2002\)](#), and has been successfully applied in the related literature.¹⁸

REMARK 1. Assumption 3 implies that, in general, the transition cannot be deterministic. Indeed, supermodularity and increasing differences of the integrand $\int f(t')q(dt'|t, a, \mu)$ must hold for *any* integrable and monotone function f , which is generally not satisfied for deterministic transitions.¹⁹ However, if $A \subseteq \mathbb{R}$ (an important special case in the applied literature) then, for any continuous and increasing function $g : A \rightarrow T$, the deterministic transition given by $q(S|t, a, \mu) = 1$, if $g(a) \in S$, and $q(S|t, a, \mu) = 0$ otherwise, satisfies our assumption.

REMARK 2. Whenever the action space A is one-dimensional and the transition function q depends only on action a , our results remain true even if the payoff function r is *not* increasing in t and the correspondence $\tilde{A}$ is *not* increasing in t in the set inclusion order. This follows directly from our constructive argument in Section 2.3 and will become clear in the remainder of the paper.

An important feature of our framework is that the sequential problem in (2) admits a recursive representation. Let $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ determine the next period distribution $\Phi(\mu)$ over types and actions in the population, given the current distribution μ . By (1), the marginal of $\Phi(\mu)$ over types in T is $\phi(\mu)(S) = \int_{T \times A} q(S|t, a, \mu)\mu(dt \times da)$, for any measurable set S . Denote $\mu_T := \text{marg}_T(\mu)$, where $\text{marg}_T(\mu)$ returns the marginal of μ on T . In the remainder of this section, we show that for any initial distribution μ and any function Φ , the value corresponding to the problem (2) satisfies

$$v^*(t, \mu; \Phi) = \max_{a \in A(t, \mu_T)} \left\{ (1 - \beta)r(t, a, \mu) + \beta \int_T v^*(t', \Phi(\mu); \Phi)q(dt'|t, a, \mu) \right\}.^{20} \quad (3)$$

Given the initial distribution μ and the perceived law of motion Φ , the player's problem is a MDP with uncertainty about the future private signal t only. Thus, under the exact law of large numbers, the sequence of aggregate distributions $\{\mu_n\}_{n \in \mathbb{N}}$ is *deterministic*. Using standard arguments, we show that the best response correspondence of each player is Markovian on the natural state space of current types t and measures μ . However, our definition of equilibrium requires consistency between such a policy and the perceived law of motion Φ . Since Φ specifies *beliefs* of players on continuation paths of

¹⁸For example, see Balbus, Refetti, and Woźny (2013) for a discussion on the nature of these assumptions.

¹⁹Indeed, the (deterministic) transition $q(S|t, a, \mu) = 1$ if $g(t, a, \mu) \in S$, and $q(S|t, a, \mu) = 0$ otherwise, does not satisfy our assumption, even when g is increasing in all variables, supermodular in a , and has increasing differences in $(a, (t, \mu))$ and (t, μ) . It is so, even when such a deterministic transition is extended by, for example, an i.d.d., multiplicative noise π over Z . In such case, the function $\int_T f(t''|t, a, \mu) = \int_Z f(z'g(t, a, \mu))\pi(dz')$ does not have increasing differences between (a, t) , unless f is convex.

²⁰Equivalently, one may use t, τ as state variables and construct μ by composing τ and a strategy $\sigma : T \rightarrow A$. In such a case, the strategy σ would have to be included as an additional parameter of the value function.

the game, we write $v^*(t, \mu; \Phi)$ to stress that the value function and the corresponding policy depend on the beliefs.²¹

2.2 Markov stationary Nash distributional equilibria

We now define the notion of equilibrium we use in this paper.

DEFINITION 1 (Markov stationary Nash distributional equilibrium). A pair (μ^*, Φ^*) with $\mu^* \in \mathcal{M}$ and $\Phi^* : \mathcal{M} \rightarrow \mathcal{M}$ is a *Markov Stationary Nash Distribution Equilibrium* (MSNDE) whenever:

- (i) there is a function v^* such that, for any $\mu \in \mathcal{M}$ and almost every $t \in T$,

$$v^*(t, \mu; \Phi^*) = \max_{a \in \bar{A}(t, \mu_T)} \left\{ (1 - \beta)r(t, a, \mu) + \beta \int_T v^*(t', \Phi^*(\mu); \Phi^*) q(dt' | t, a, \mu) \right\};$$

- (ii) there is a measurable selection σ_{μ, Φ^*} of the correspondence $\Sigma_{\mu, \Phi^*} : T \rightrightarrows A$:

$$\Sigma_{\mu, \Phi^*}(t) := \arg \max_{a \in \bar{A}(t, \mu_T)} \left\{ (1 - \beta)r(t, a, \mu) + \beta \int_T v^*(t', \Phi^*(\mu); \Phi^*) q(dt' | t, a, \mu) \right\},$$

$$\mu^* = \mu_T^*(\text{id}_T, \sigma_{\mu^*, \Phi^*})^{-1} \text{ and } \Phi^*(\mu) = \phi(\mu)(\text{id}_T, \sigma_{\Phi^*(\mu), \Phi^*})^{-1}, \text{ for any } \mu \in \mathcal{M}.$$

An MSNDE consists of an initial distribution μ^* on types and actions, a Markov transition function Φ^* , and it involves an equilibrium policy $\sigma_{\mu, \Phi^*} : T \rightarrow A$. MSNDE is stationary in the sense that equilibrium strategies and beliefs of all players are time invariant. Nevertheless, the continuation payoff $v^*(\cdot, \Phi^*(\mu^*); \Phi^*)$ implies a *dynamic* interaction of each player with the future distributions of types and actions in the population, through the equilibrium law of motion Φ^* .

Condition (i) is a standard Bellman equation characterizing players' best reply correspondences. The second part of the definition imposes a two-fold consistency. First of all, $\mu^* = \mu_T^*(\text{id}_T, \sigma_{\mu^*, \Phi^*})^{-1}$ guarantees that the distribution of actions is generated by the equilibrium strategy σ_{μ^*, Φ^*} , given the initial distribution of types and the equilibrium law of motion. In addition, we require that $\Phi^*(\mu) = \phi(\mu)(\text{id}_T, \sigma_{\Phi^*(\mu), \Phi^*})^{-1}$. Thus the perceived macro belief and the actual law of motion for aggregate distributions (i.e., generated by the best-response selection $\sigma_{\Phi^*(\mu), \Phi^*}$) must coincide.²² The Markov transition Φ^* specifies common beliefs each player uses to determine future paths of equilibrium distributions. In macroeconomic literature on recursive equilibria, such beliefs are often called *rational*. Since $\Phi^*(\mu) = \phi(\mu)(\text{id}_T, \sigma_{\Phi^*(\mu), \Phi^*})^{-1}$, for any $\mu \in \mathcal{M}$, the function Φ^* specifies beliefs “on” and “off” equilibrium paths.

To state our main result, we introduce one final piece of notation. Let

$$\mathcal{D} := \{ \Phi : \mathcal{M} \rightarrow \mathcal{M} : \Phi \text{ is increasing and monotone} \\ \text{inf-preserving and } \text{marg}_T(\Phi(\mu)) = \phi(\mu), \text{ for any } \mu \in \mathcal{M} \}. \quad (4)$$

²¹Compare with the equilibrium in Kalai and Shmaya (2018) for large but finite repeated games.

²²Since we work with no aggregate uncertainty, we do not require that Φ^* is measurable.

Therefore, we restrict our attention to functions/beliefs Φ that are increasing and monotone inf-preserving. We endow $\mathcal{D}$ with the pointwise order, that is, function Φ' dominates Φ if the probability measure $\Phi'(\mu)$ first order stochastically dominates $\Phi(\mu)$, for all $\mu \in \mathcal{M}$. We endow $\mathcal{D}$ with the topology of pointwise convergence.

REMARK 3. Dually, let $\mathcal{D}' := \{\Phi : \mathcal{M} \rightarrow \mathcal{M} : \Phi \text{ increasing and monotone } \sup\text{-preserving and } \text{marg}_T(\Phi(\mu)) = \phi(\mu), \text{ for any } \mu \in \mathcal{M}\}$. To save space, we focus on $\mathcal{D}$, but all constructions and results have their counterpart in $\mathcal{D}'$.

Consider the main result of this paper.

THEOREM 1. *Let Assumptions 1–4 be satisfied. Then any large stochastic game with complementarities has a Markov Stationary Nash Distributional Equilibrium (MSNDE). In particular, there exists a greatest MSNDE of the game in $\mathcal{M} \times \mathcal{D}$ and a least MSNDE of the game in $\mathcal{M} \times \mathcal{D}'$.*

We postpone the proof until Section 2.3. The above theorem requires some comment. First, apart from providing sufficient conditions for existence of an MSNDE, Theorem 1 guarantees existence of a *monotone* MSNDE, consisting of monotone beliefs Φ^* and monotone strategies σ_{μ, Φ^*} . Moreover, the space of all MSNDE in $\mathcal{M} \times \mathcal{D}$ admits a greatest element. Similarly, there exists a *least* MSNDE in the space $\mathcal{M} \times \mathcal{D}'$. Finally, if the set of maximizers to the optimization problem on the right-hand side in (3) is unique, then the set of MSNDE is *chain complete*, that is, closed under monotone sequences of equilibria in $\mathcal{M} \times \{\mathcal{D} \cap \mathcal{D}'\}$.

REMARK 4. Any MSNDE induces a *sequential distributional equilibrium* as in Jovanovic and Rosenthal (1988), that is, $\{\mu_n^*\}_{n \in \mathbb{N}}$, where $\mu_1^* = \mu^*$ and $\mu_n^* = \Phi^*(\mu_{n-1}^*)$. In fact, such sequential distributional equilibrium can be constructed for any initial distribution τ_1 of types of all players. Indeed, it is clear from Definition 1 that μ^* can be constructed using any distribution on types τ_1 and a stationary equilibrium policy σ^* .²³

A natural question is whether there is an invariant distribution induced by MSNDE that would give rise to a *stochastic stationary equilibrium*—a fixed point of a *global* distributional equilibrium dynamics generated by an MSNDE.

PROPOSITION 1 (Invariant distributions). *Under Assumptions 1–4, there exists a stochastic stationary equilibrium. In particular, there exists a greatest invariant distribution $\bar{\nu}$ induced by the greatest MSNDE $(\bar{\mu}^*, \bar{\Phi}^*)$, that is, $\bar{\nu} = \bar{\Phi}^*(\bar{\nu})$ and a least invariant distribution $\underline{\nu}$ induced by the least MSNDE $(\underline{\mu}^*, \underline{\Phi}^*)$.*

We omit the proof. The existence of invariant distributions is guaranteed by monotonicity of $\bar{\Phi}$ and $\underline{\Phi}$, and the fact that the space of measures $\mathcal{M}$ is a chain complete poset.²⁴ By Proposition A.1 in the Appendix, both operators admit a greatest and a least

²³See also our construction in equations (6) and (7).

²⁴See, for example, Lemma 2 in Balbus, Dziewulski, Reffett, and Woźny (2019).

fixed point (invariant distribution), and these extremal fixed points can be obtained through successive iterations on the mappings $\overline{\Phi}^*$ and $\underline{\Phi}^*$, respectively. Additionally, for any MSNDE (μ^*, Φ^*) , the pair $(\Phi^*(\mu^*), \Phi^*)$ is also an MSNDE. In fact, the pair (ν, Φ^*) is a MSNDE as well, for any invariant distribution ν generated by Φ^* .²⁵

We prove Theorem 1 in the following subsection. It is important to point out that our argument is *constructive* and based upon *order continuous* successive approximation techniques. In particular, we introduce an *explicit* iterative algorithm that approximates the *greatest* equilibrium of the game. To present our construction, we require additional notation. For any $\mu \in \mathcal{M}$, $\Phi \in \mathcal{D}$, and function v , let

$$\Gamma(t, \mu, \Phi; v) := \arg \max_{a \in \tilde{A}(t, \mu_T)} \left\{ (1 - \beta)r(t, a, \mu) + \beta \int_T v(t', \Phi(\mu), \Phi) q(dt'|t, a, \mu) \right\}, \quad (5)$$

that is, the set of maximizers of the player's MDP. Define a greatest element of the set by $\overline{\gamma}(t, \mu, \Phi; v)$, if it exists. Let $\star$ be a binary operation between $\tau \in \mathcal{M}_T$ and the set of measurable functions $h: T \rightarrow \mathcal{A}$ returning probability measure on $T \times \mathcal{A}$

$$\tau \star h := \tau(\text{id}_T, h)^{-1}. \quad (6)$$

Define the operator $\overline{\Psi}$ mapping $\mathcal{M} \times \mathcal{D}$ into itself, where $\overline{\Psi}(\mu, \Phi) = (\mu', \Phi')$ and

$$\mu' := \mu_T \star \overline{\gamma}(\cdot, \mu, \Phi; v^*) \quad \text{and} \quad \Phi'(\mu) := \phi(\nu) \star \overline{\gamma}(\cdot, \Phi(\mu), \Phi; v^*), \quad (7)$$

for all $\mu \in \mathcal{M}$, where $v^*: T \times \mathcal{M} \times \mathcal{D} \rightarrow \mathbb{R}$ is a function solving (3).

PROPOSITION 2 (Bounds approximation). *Let $\bar{\mu}$ and $\bar{\Phi}$ be the greatest elements of $\mathcal{M}$ and $\mathcal{D}$, respectively. Under Assumptions 1–4, $\lim_{n \rightarrow \infty} \overline{\Psi}^n(\bar{\mu}, \bar{\Phi}) = (\bar{\mu}^*, \bar{\Phi}^*)$, where $(\bar{\mu}^*, \bar{\Phi}^*)$ is the greatest MSNDE.*

An analogous approximation result holds for the least MSNDE. Our results contribute to several strands of economics literature. First, we extend the results on existence of equilibria in large anonymous sequential games that date back to Jovanovic and Rosenthal (1988), Bergin and Bernhardt (1992), and Karatzas, Shubik, and Sudderth (1994). In particular, we prove existence of *minimal state space stationary Markovian* distributional Nash equilibrium within the subclass of games that (in addition) possess strategic complementarities. Second, we extend the class of games with strategic complementarities (GSC) to a dynamic setting with a measure space of players. Following Van Zandt (2010) and Van Zandt and Vives (2007), a few recent papers generalized the class of supermodular games and GSC to normal-form games with complete and incomplete information. See, for example, Balbus, Reffett, and Woźny (2015a), Balbus, Dziewulski, Reffett, and Woźny (2015, 2019), and Bilancini and Boncinelli (2016), who construct the necessary tools for a study of large static games with strategic complementarities. We extend these results to Markovian equilibria in large *dynamic* games.

²⁵However, it must be that $\bar{\nu}$ is dominated by $\bar{\mu}^*$.

The tools used in our paper extend significantly the ones in Balbus, Reffett, and Woźny (2013, 2014), where the question under study is the existence and characterization of Markovian equilibria in dynamic games with a *finite* number of players. Specifically, within the context of a large game framework, the current paper relaxes some strong geometrical assumptions on the (aggregate) transition function q required in stochastic supermodular games with a finite number of players.²⁶ Indeed, the context of a large game allows us to avoid multiple issues related to extensive-form supermodular games as discussed, for example, in Amir (2002), Echenique (2004), Vives (2009), and Mensch (2020). Our assumptions guarantee that the stationary value functions in each player's decision problem has increasing differences in the private type and the distribution over types and actions in the population. With this structure in place, our large stochastic supermodular game remains extensive-form supermodular over the *infinite horizon*. This is critical for equilibrium comparative statics/dynamics. Given the distributional specification of the game, we are able to avoid issues with characterizing dynamic complementarities in actions across periods and beliefs, reported recently in Mensch (2020) for dynamic Bayesian games with a finite number of players. Finally, as in Balbus, Reffett, and Woźny (2014), the existence of the extremal MSNDEs is proven constructively, by applying successive approximations starting from the greatest (resp., the lowest) strategies. In this sense, we provide the applied researchers with tools that allow to approximate sequences of equilibrium distributions.

2.3 Construction of equilibria

We devote this subsection to the proof of Theorem 1. We present our argument via a number of lemmas, each of which may be of individual interest. We prove existence of a greatest MSNDE only. The argument for a least MSNDE is analogous.

Let Assumptions 1–4 be satisfied. We begin by showing that the problem in (2) admits a recursive representation. In particular, for any Markov transition function $\Phi \in \mathcal{D}$, there is a unique function v satisfying (3). Let $\mathcal{V}$ be the space of functions $v: T \times \mathcal{M} \times \mathcal{D} \mapsto \mathbb{R}$ such that: (i) v is uniformly bounded by a value $\bar{r} > 0$, (ii) $v(\cdot, \mu, \Phi)$ is increasing and continuous, for any $(\mu, \Phi) \in \mathcal{M} \times \mathcal{D}$, (iii) $v(t, \cdot, \cdot)$ is monotone inf-preserving, for any $t \in T$, and (iv) v has increasing differences in $(t, (\mu, \Phi))$. We endow $\mathcal{V}$ with the sup-norm topology $\|\cdot\|_\infty$.²⁷

LEMMA 1. $\mathcal{V}$ is a complete metric space.

Given that $\mathcal{V}$ is a subset of all bounded functions, it is a subset of a Banach space. Hence, it suffices to show that the set is closed. Since continuity, monotonicity, and increasing differences are preserved in the sup-norm convergence, the main difficulty is to show that any limit of monotone inf-preserving functions preserves this property. We prove this in the Appendix.

²⁶See Assumption 1 in Balbus, Reffett, and Woźny (2013) and Assumption 2 in Balbus, Reffett, and Woźny (2014), which are necessary to guarantee that equilibrium dynamics do not lead to an absorbing state.

²⁷That is, we have $\|v\|_\infty := \sup_{(t, \mu, \Phi) \in T \times \mathcal{M} \times \mathcal{D}} |v(t, \mu; \Phi)|$.

The next lemma provides an important feature of Markov transition functions Φ . It follows immediately from Lemma A.2 in the [Appendix](#).

LEMMA 2. *Let $\{\mu_k\}_{k \in \mathbb{N}}$ be a decreasing sequence in $\mathcal{M}$ that weakly converges to μ in $\mathcal{M}$. Let $\{\Phi_k\}_{k \in \mathbb{N}}$ be a decreasing sequence in $\mathcal{D}$ that pointwise weakly converges to some Φ in $\mathcal{D}$. Then $\{\Phi_k(\mu_k)\}_{k \in \mathbb{N}}$ weakly converges to $\Phi(\mu)$.*

Define an operator $B : \mathcal{V} \rightarrow \mathcal{V}$ as

$$(Bv)(t, \mu, \Phi) := \max_{a \in \tilde{A}(t, \mu_T)} \left\{ (1 - \beta)r(t, a, \mu) + \beta \int_T v(t', \Phi(\mu), \Phi) q(dt' | t, a, \mu) \right\}. \quad (8)$$

Some basic properties of the operator B are provided below.

LEMMA 3. *For any $v \in \mathcal{V}$, function (Bv) is continuous and increasing in t , jointly monotone inf-preserving in (μ, Φ) , and has increasing differences in $(t, (\mu, \Phi))$.*

Denote the function within the brackets in (8) by

$$F(t, a, \mu; v, \Phi) := (1 - \beta)r(t, a, \mu) + \beta \int_T v(t', \Phi(\mu), \Phi) q(dt' | t, a, \mu). \quad (9)$$

Given Assumptions 2–4, $F(t, a, \mu; v, \Phi)$ is increasing in t , jointly continuous in (t, a) , and has increasing differences in $(a, (t, \mu, \Phi))$ and $(t, (\mu, \Phi))$. We claim it is monotone inf-preserving in (μ, Φ) . Suppose that $\{(\mu_n, \Phi_n)\}_{n \in \mathbb{N}}$ is a decreasing sequence that converges to (μ, Φ) . By Lemma 2, we have $\Phi_n(\mu_n) \rightarrow \Phi(\mu)$. By Assumption 2 and the choice of the set $\mathcal{V}$, it must be that both $r(t, a, \mu_k) \rightarrow r(t, a, \mu)$ and $v(t, \Phi_k(\mu_k), \mu_k) \rightarrow v(t, \Phi(\mu), \mu)$. Moreover, we have $\int_T v(t', \Phi_k(\mu_k), \mu_k) q(dt' | t, a, \mu_k) \rightarrow \int_T v(t', \Phi(\mu), \mu) q(dt' | t, a, \mu)$, which follows from Lemma A.3 in the [Appendix](#). Hence, F is inf-preserving. We are ready to prove Lemma 3.

PROOF OF LEMMA 3. Continuity of (Bv) follows from Berge's maximum theorem (see Theorem 17.31 in [Aliprantis and Border \(2006\)](#)). Monotonicity of (Bv) in t is implied by monotonicity of F and the fact that $\tilde{A}$ increases in t with respect to set inclusion. To show that it is monotone inf-preserving in (μ, Φ) , take any decreasing sequence $\{(\mu_k, \Phi_k)\}_{k \in \mathbb{N}}$ that converges to some (μ, Φ) . We know that $F(t, a_k, \mu_k; v, \Phi_k) \rightarrow F(t, a, \mu; v, \Phi)$ whenever $a_k \rightarrow a$. By Lemma A.4 in the [Appendix](#), this suffices for $(Bv)(t, \mu_k, \Phi_k) \rightarrow (Bv)(t, \mu, \Phi)$. Finally, we show that (Bv) has increasing differences in $(t, (\mu, \Phi))$, for any $v \in \mathcal{V}$. Equivalently, we claim that $w(t, \mu; v, \Phi) = \max_{a \in \tilde{A}(t, \mu_T)} F(t, a, \mu; v, \Phi)$ has increasing differences in $(t, (\mu, \Phi))$, for any v . By Assumptions 2 and 3, the function $F(t, a, \mu; v, \Phi)$ is supermodular in a , has increasing differences in $(a, (t, \mu, \Phi))$ and in $(t, (\mu, \Phi))$. The rest follows from Lemma A.1 in the [Appendix](#) that generalizes Lemma 1 in [Hopenhayn and Prescott \(1992\)](#). $\square$

PROPOSITION 3. *The operator $B : \mathcal{V} \rightarrow \mathcal{V}$ has a unique fixed point in $\mathcal{V}$.*

Indeed, Lemma 3 guarantees that B is a well-defined operator that maps a complete metric space $\mathcal{V}$ into itself (recall Lemmas 1 and 3). Therefore, it suffices to show that B is a contraction mapping on $\mathcal{V}$. This fact follows from Blackwell's sufficient conditions for contraction and can be shown using an argument analogous to the one supporting Theorem 3.3 in [Stokey, Lucas, and Prescott \(1989\)](#).²⁸ Thus B has a unique fixed point. Finally, showing that the value coincides with the value of the original problem (2) can be done using standard arguments.²⁹

Given that the value function has increasing differences in both arguments (t, μ) and the transition Φ^* is monotone, we can guarantee that the current actions of players and their beliefs regarding the future distribution of types and actions in the population are complements. As a result, we are able to show existence of equilibrium by applying constructive order-theoretic tools (rather than purely topological constructions), and this allows us to unify our existence results with our subsequent equilibrium comparative dynamics results.

We proceed with the second-half of the argument where we prove existence of a greatest MSNDE. Recall the definition of the correspondence Γ from (5), with its greatest selection $\bar{\gamma}: T \times \mathcal{M} \times \mathcal{D} \rightarrow A$. Consider the following lemma.

LEMMA 4. *For any $v \in \mathcal{V}$, the greatest selection $\bar{\gamma}$ is a well-defined function, measurable in t , increasing in (t, μ, Φ) , and monotone inf-preserving.*

PROOF. For any $v \in \mathcal{V}$, we have $\Gamma(t, \mu; v, \Phi) = \arg \max_{a \in \tilde{A}(t, \mu_T)} F(t, a, \mu; v, \Phi)$. It is straightforward to verify that F is supermodular and continuous in a . Since the set $\tilde{A}(t, \mu_T)$ is a complete sublattice of A , by Corollary 4.1 in [Topkis \(1978\)](#), the set $\Gamma(t, \mu; v, \Phi)$ is a complete sublattice of A . Therefore, it admits both a greatest and a least element. We show that $\bar{\gamma}$ is measurable in the [Appendix](#). Monotonicity follows from increasing differences of F and Theorem 6.2 in [Topkis \(1978\)](#).³⁰ To show that $\bar{\gamma}$ is monotone inf-preserving, let $\{(\mu_k, \Phi_k)\}_{k \in \mathbb{N}}$ be a decreasing sequence converging to (μ, Φ) . By the previous argument, sequence $\{\bar{\gamma}(t, \mu_k, \Phi_k; v)\}_{k \in \mathbb{N}}$ is decreasing. Suppose it converges to some γ , and thus $\bar{\gamma}(t, \mu_k, \Phi_k; v) \geq \gamma$, for all $k \in \mathbb{N}$. Since F is continuous and monotone inf-preserving, Lemma A.4 in the [Appendix](#) guarantees that $\gamma \in \Gamma(t, \mu; v, \Phi)$. Thus it must be that $\gamma \leq \bar{\gamma}(t, \mu, \Phi; v)$, and so $\gamma \leq \bar{\gamma}(t, \mu, \Phi; v) \leq \bar{\gamma}(t, \mu_k, \Phi_k; v)$. $\square$

For the following lemma, recall the definition of the binary operation $\star$ in (6).

LEMMA 5. *For any measures $\tau, \tau' \in \mathcal{M}_T$ such that τ' first order stochastically dominates τ , and any increasing functions $h, h': T \times A \rightarrow A$ such that h' dominates h pointwise, the measure $(\tau' \star h')$ first order stochastically dominates $(\tau \star h)$.*

The proof of the above lemma is straightforward; hence, we omit it.

²⁸Indeed, if v' pointwise dominates v , then Bv' pointwise dominates Bv . Moreover, for any $v \in \mathcal{V}$ and constant $a \geq 0$, we have $[B(v + a)](t, \mu; \Phi) \leq (Bv)((t, \mu; \Phi) + \beta a)$.

²⁹See, for example, Theorem 9.2 in [Stokey, Lucas, and Prescott \(1989\)](#).

³⁰See the glossary in the [Appendix](#).

LEMMA 6. Let $\{\tau_k\}_{k \in \mathbb{N}}$ be a decreasing sequence in $\mathcal{M}_T$ converging to some τ , and let $\{h_k\}_{k \in \mathbb{N}}$ be a (pointwise) decreasing sequence converging to some h , where $h_k : T \times A \rightarrow A$ are increasing and monotone inf-preserving functions. Then $(\tau_k \star h_k) \rightarrow (\tau \star h)$ in the sense of weak convergence.

PROOF. This follows from Lemma A.2 in the Appendix.³¹ We only need to show that $\tau \star h$ is inf-preserving in h . Take an arbitrary $\tau \in \mathcal{M}_T$ and let h_k be a decreasing sequence of Borel functions mapping T to A . Let $h = \lim_{k \rightarrow \infty} h_k$. Then, for any measurable, continuous, and bounded function $f : T \times A \rightarrow \mathbb{R}$,

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{T \times A} f(t, a)(\tau \star h_k)(dt \times da) &= \lim_{k \rightarrow \infty} \int_T f(t, h_k(t))\tau(dt) \\ &= \int_T f(t, h(t))\tau(dt) = \int_{T \times A} f(t, a)(\tau \star h)(dt \times da). \end{aligned}$$

Hence, $(\tau \star h_k) \rightarrow (\tau \star h)$ weakly. This completes the proof. $\square$

Take the unique function v^* that solves the equation (3). Define operator $\bar{\Psi}$ as in (7). Given monotonicity of $\bar{\gamma}(t, \mu, \Phi; v)$ and Lemma 5, we conclude that it is increasing. Moreover, by Lemmas 4 and 6, it is also monotone inf-preserving. Before completing the proof of Theorem 1, we require one more ancillary result.

LEMMA 7. The set $\mathcal{D}$ is a lower chain complete poset.

PROOF. Let $\{\Phi_j\}_{j \in J}$ be a chain in $\mathcal{D}$ and $\Phi := \bigwedge_{j \in J} \Phi_j$. It suffices to show that Φ is monotone inf-preserving. Let $\{\mu_k\}_{k \in \mathbb{N}}$ be a decreasing sequence in $\mathcal{M}$ that converges to μ . For any k, j , and an increasing, measurable function $f : T \times A \rightarrow \mathbb{R}$,

$$\int_{T \times A} f(t, a)(\Phi\mu)(dt \times da) \leq \int_{T \times A} f(t, a)(\Phi\mu_k)(dt \times da) \leq \int_{T \times A} f(t, a)(\Phi_j\mu_k)(dt \times da).$$

As $k \rightarrow \infty$, we obtain

$$\begin{aligned} \int_{T \times A} f(t, a)(\Phi\mu)(dt \times da) &\leq \liminf_{k \rightarrow \infty} \int_{T \times A} f(t, a)(\Phi\mu_k)(dt \times da) \\ &\leq \limsup_{k \rightarrow \infty} \int_{T \times A} f(t, a)(\Phi\mu_k)(dt \times da) \\ &= \int_{T \times A} f(t, a)(\Phi_j\mu)(dt \times da). \end{aligned}$$

We conclude by taking the infimum with respect to j on the right-hand side. $\square$

We proceed with the proof of Theorem 1.

³¹Here, the role of Ξ plays $\mathcal{M}_T$, and the role of f_k plays $h \mapsto (\tau_k \star h)$.

PROOF OF THEOREM 1. It suffices to show that there is a greatest fixed point of $\bar{\Psi}$ defined in (7). Note that $\bar{\Psi}$ is monotone in (μ, Φ) . Indeed, by Lemma 4, $\bar{\gamma}(t, \mu, \Phi; v^*)$ is jointly increasing in (t, μ, Φ) . By Lemma 6, this implies monotonicity of μ' in (7). By the same argument Φ' in (7) is increasing in μ and Φ . Moreover, by Lemmas 4 and 6, we conclude that $\bar{\Psi}$ is a monotone inf-preserving self-map on $\mathcal{M} \times \mathcal{D}$. By applying Proposition A.1 in the Appendix, we conclude that there exists a greatest MSNDE.³² $\square$

3. MONOTONE EQUILIBRIUM COMPARATIVE DYNAMICS

Here, we discuss the nature of constructive monotone equilibrium comparative dynamics (see, e.g., Huggett (2003)) in the class of games studied in Section 2. To do this, we parameterize primitives of our game with a parameter θ , that belongs to a poset $(\Theta, \geq_\Theta)$, and seek conditions under which MSNDEs are ordered with respect to θ . Given our definition of equilibrium, this means that the selection $\theta \rightarrow \mu_\theta^*$ and the equilibrium law of motion $\theta \rightarrow \Phi_\theta^*$ are increasing in the following sense: if $\theta' \geq_\Theta \theta$, then $\mu_{\theta'}^*$ stochastically dominates μ_θ^* , and the measure $\Phi_{\theta'}^*(\mu)$ stochastically dominates $\Phi_\theta^*(\mu)$, for all measures $\mu \in \mathcal{M}$. Since our notion of equilibrium is inherently dynamic, we use the term monotone comparative *dynamics* rather than comparative statics. We define a positive shock.³³

ASSUMPTION 5 (Positive shock). *Let Θ be a poset. (i) The payoff function $r(t, a, \mu; \theta)$ has increasing differences in (a, θ) and (t, θ) ; (ii) the transition kernel $q(\cdot|t, a, \mu; \theta)$ is increasing in θ and has increasing differences in (a, θ) and (t, θ) ; (iii) the feasible action correspondence $\tilde{A}(t, \mu; \theta)$ has strict complementarities in (t, θ) .*

THEOREM 2 (Monotone comparative distributional dynamics). *Suppose that the parameterized mappings $r(\cdot, \theta)$, $q(\cdot; \theta)$, and $\tilde{A}(\cdot; \theta)$ satisfy Assumptions 1–4, for all $\theta \in \Theta$. Under Assumption 5, both the greatest equilibrium $(\bar{\mu}_\theta^*, \bar{\Phi}_\theta^*)$ and the least equilibrium $(\underline{\mu}_\theta^*, \underline{\Phi}_\theta^*)$ is increasing in θ .*

PROOF. We prove the case for the greatest equilibrium only. The proof for the lowest equilibrium is analogous. Let $\bar{\Psi}_\theta$ be the counterpart of the operator $\bar{\Psi}$ in the parameterized game with $\theta \in \Theta$. Similarly, we define ϕ_θ , Φ_θ , and $\bar{\gamma}_\theta$, mutatis mutandis. Given that $q(\cdot|t, a, \mu; \theta)$ is increasing in θ , it suffices to show that $\theta \rightarrow \bar{\gamma}_\theta$ is increasing. Under our assumptions, the objective $(1 - \beta)r(t, a, \mu, \theta) + \beta \int_{\mathcal{T}} v^*(t', \Phi_\theta(\mu), \Phi_\theta, \theta) q(dt'|t, a, \mu, \theta)$ has increasing differences in (a, θ) , and the function $v^*(t', \Phi_\theta(\mu), \Phi_\theta, \theta)$ has increasing differences in (t, θ) , for any $\mu \in \mathcal{M}$ (recall Lemma A.1 in the Appendix and the argument in the second part of the proof to Lemma 3). By Theorem 6.2 in Topkis (1978) (see the

³²Whenever the best response γ is single valued, it can be shown using Theorem 9 in Markowsky (1976) that the fixed-point set of $\bar{\Psi} = \Psi$ is a chain complete poset.

³³Our notion of a positive shock is consistent with the terminology of Acemoglu and Jensen (2015). However, we consider comparative equilibrium *transitional* dynamics. In this sense, our question is closely related to the issues studied in Huggett (1997). Recall, in Huggett (1997), monotone aggregate dynamics are only available from initial distributions below the stochastic steady state. Here, our MSNDE dynamics are globally monotone.

glossary in the [Appendix](#)), we conclude that $\bar{\gamma}_\theta$ is increasing in θ . By Assumption 5, we know that $\theta \rightarrow \phi^\theta$ is increasing. By Lemma 5, the same property is inherited by $\bar{\Psi}_\theta$, that is, for $\theta' \geq \theta$, $\bar{\Psi}_{\theta'}(\mu, \Phi) = (\mu_{\theta'}, \Phi_{\theta'})$ pointwise dominates $\bar{\Psi}_\theta(\mu, \Phi) = (\mu_\theta, \Phi_\theta)$. Equivalently, $\mu_{\theta'}$ first order stochastically dominates μ_θ and the measure $\Phi_{\theta'}(\mu)$ first order stochastically dominates $\Phi_\theta(\mu)$, for all $\mu \in \mathcal{M}$. As in the proof of Theorem 1, $\bar{\Psi}_\theta$ is an increasing operator, for any fixed θ . To complete the proof we apply Proposition A.2, recalling that the poset of distributions and the poset of uniformly bounded functions are chain complete. $\square$

An immediate corollary follows: Under Assumptions 1–4, the greatest equilibrium increases in the initial distribution of types τ_1 .³⁴ Indeed, if we let $\theta = \tau_1$ and $\Theta = \mathcal{M}_T$ is ordered in the stochastic sense, then Assumption 5 holds.

Our monotone comparative dynamics results both improve upon and complement the results in the existing literature, including those found in [Adlakha and Johari \(2013\)](#), [Acemoglu and Jensen \(2010, 2015\)](#), [Light and Weintraub \(2021\)](#). These papers discuss comparative statics of *stochastic steady-state equilibria* or *mean-field equilibria*, rather than MSNDE. In particular, they focus on (a) the set of equilibrium invariant and/or steady-state distributions, and (b) games with an aggregative structure. In contrast, we provide conditions under which (extremal) MSNDE equilibrium transition paths exist and are increasing *globally* in the deep parameters. In our case, as the equilibrium distribution μ_θ^* and the law of motion/belief Φ_θ^* increase in θ , so does the distribution $\Phi_\theta^*(\mu_\theta^*)$ in the following period, and so on.³⁵ Therefore, the *entire equilibrium path* shifts with respect to the parameter θ . Further, as these equilibrium paths converge to a steady state, a greatest invariant distribution $\bar{\nu}_\theta$ induced by the greatest equilibrium is also increasing in θ .³⁶

It bears mentioning that the assumptions in [Acemoglu and Jensen \(2015, 2018\)](#), and [Light and Weintraub \(2021\)](#) are not sufficient for an analogous monotone comparative dynamics result. The key difference between these works and ours is that, when studying comparative statics of a steady-state equilibrium (or mean-field equilibrium), one does *not* investigate dynamic complementarities between current actions and future distributions of types and actions in the population. This is because stochastic steady-state equilibria are inherently static, unlike our dynamic notion of MSNDE. For this reason, we require extra assumptions to guarantee that the equilibrium dynamics of the economy under study are sufficiently monotone. These assumptions are not vacuous, as the literature includes many examples of dynamic economies that do *not* satisfy our conditions. Indeed, the standard Bewley–Huggett–Aiyagari models of wealth distribution with infinitely lived agents and the presence of incomplete markets are, in general, not monotone in the sense specified here. We address this in Section 4.6.

Our results apply to distributions over the *multidimensional* space $\mathbb{R}^n$. In fact, the multidimensionality is inherent if one studies distributions over types *and* actions (as

³⁴Analogous comparative dynamics apply to the least MSNDE.

³⁵This complements the approach to transitional dynamics in [Huggett \(1997\)](#).

³⁶A similar argument applies to the least equilibrium global distributional dynamic path and the least invariant distribution (or the least steady state).

in our motivating example). Since spaces of measures over multidimensional spaces are *not* lattices, we employ the alternative tool from Proposition A.2 in the [Appendix](#), as the well-known Tarski's fixed-point theorem does *not* apply.³⁷

Finally, our monotone comparative statics/dynamics results are *constructive*. We characterize the chain of parameterized equilibria converging to the one of interest, for a particular parameter θ . This is of utmost importance for applied economists who calibrate moments of equilibrium invariant distributions, or econometrically estimate equilibrium comparative statics/dynamics in the data (e.g., with quantile methods in [Echenique and Komunjer \(2009, 2013\)](#)).³⁸

4. APPLICATIONS AND EXAMPLES

Here, we discuss some economic applications of our results.

4.1 Motivating example revisited

Recall the motivating example from the [Introduction](#). In each period, the type of a player is determined by their level of capital/wealth $t \in T = [0, 1]$. Their actions (investments) $a \in A = [0, 1]$ are chosen from the feasible set $\tilde{A}(t, \tau) = [0, t]$. Given the distribution μ of types-actions of all players, the payoff in a single period is

$$r(t, a, \tau, \theta) := \int_{A \times T} [\theta m(t - \tilde{t}) + w(t - a - \tilde{t} + \tilde{a})] \mu(d\tilde{a} \times d\tilde{t}).$$

Here, we introduce a positive parameter θ with respect to the initial example.

Given an investment a , the cumulative probability distribution of the capital level t' in the following period is $q(t'|a)$. Thus, conditional on the macro belief Φ , the Bellman equation determining the player's value function is

$$v(t, \mu; \Phi) = \max_{a \in \tilde{A}(t, \mu_T)} \left\{ (1 - \beta)r(t, a, \mu, \theta) + \beta \int v(t', \Phi(\mu); \Phi) q(dt'|a) \right\}.$$

This game satisfies Assumptions 1–4. Indeed, correspondence $\tilde{A}$ is measurable, continuous, compact-valued, increasing in t , and satisfies strong complementarities. Given that the functions m and w are continuous, increasing, and concave, the function r is continuous over $T \times A$, increasing over T , and has increasing differences in $(a, (t, \mu))$ and (t, μ) . The function is also (trivially) supermodular in a , and continuous in μ . As long as the distribution q is continuous in a , the requirements of Theorem 1 for existence of a greatest MSNDE are satisfied.

As it was pointed out in Section 2, the equilibrium pair (μ^*, Φ^*) generates the entire equilibrium path of distributions $\{\mu_n^*\}$, where $\mu_1^* = \mu^*$ and $\mu_{n+1} = \Phi^*(\mu_n)$, which allows us to track the dynamics of the model. Moreover, the sequence converges to an invariant distribution, allowing for the study of steady states.

³⁷See also the discussion in Section 3 of [Light and Weintraub \(2021\)](#).

³⁸See [Cao, Luo, and Nie \(2020\)](#) for a discussion regarding numerical methods related to the global dynamic equilibrium models such as ours.

Apart from existence and approximation of equilibria, Theorem 2 allows us to say more about its equilibrium comparative dynamics. In particular, the equilibrium (μ^*, Φ^*) and the corresponding sequence $\{\mu_n^*\}$ increase as the initial distribution of types τ_1 increases in the first-order stochastic sense. That is, along an equilibrium path that converges to a steady state, players invest more and have higher capital levels (stochastically). In addition, the equilibrium changes monotonically with respect to the parameter θ . One can easily verify that the return function r has increasing differences in (a, θ) and (t, θ) . As the correspondence $\tilde{A}$ and transition kernel q are independent of θ , this suffices for the equilibrium path to be increasing in θ . Thus, the higher the weight of the wealth-driven status, the higher (stochastically) are the investments in the population.

These results hold for a more general transition $q(\cdot|t, a, \mu)$, that depends on the investment of each player, their type, and the distribution of wealth-investments in the population. However, this requires Assumption 3 to hold.

4.2 Dynamics of social distance

Next, we analyze a dynamic model of *social distance* based on Akerlof (1997).³⁹ Consider a measure space of agents. Let $T = [0, 1]$ be the set of all possible social positions in the population. Each period an individual is characterized by an *identity* $t \in T$ (private type), which determines the social position to which the agent aspires. Each period the agent chooses their own social position (action) $a \in A := [0, 1]$. The set of social positions feasible to the agent with identity t is $\tilde{A}(t, \tau) := [\underline{a}(t), \bar{a}(t)]$, where functions $\underline{a}, \bar{a} : T \rightarrow A$ are increasing functions and satisfy $\underline{a}(t) \leq t \leq \bar{a}(t)$, for all $t \in T$. Thus, limiting social mobility.

When choosing social position, there is a trade-off between *idealism* and *conformism*. On one hand, the individual wants the social status a to be as close to their identity t as possible. Specifically, given some continuous, decreasing, and concave function $m : [0, 1] \rightarrow \mathbb{R}$, the agent wants to maximize $m(|a - t|)$, that captures idealism. On the other hand, the player experiences discomfort when interacting with agents that have different social position from theirs. Whenever an agent of a social position a encounters an agent of a social position a' , they each receive utility $w(|a - a'|)$, for some continuous, decreasing, and concave function $w : [0, 1] \rightarrow \mathbb{R}$. This summarizes conformism.

Suppose that $\nu(t'|t)$ is a cumulative probability distribution determining the likelihood of an agent with identity t meeting someone with identity t' . We assume it is continuous and first order stochastically increasing in t . It captures the idea that *similar minds think alike* and players with similar identity are more likely to meet. Given the distribution of types-actions μ , the within-period payoff of an agent of an identity t and

³⁹The model is related to multiple strands of the social economics literature, including models of identity and economic choice as in Akerlof and Kranton (2000), or models with endogenous social reference points, including Bernheim (1994), Brock and Durlauf (2001), Bisin, Moro, and Topa (2011), and Blume et al. (2015). This example extends the static model in Balbus et al. (2019).

a social position a is

$$r(t, a, \mu) := m(|a - t|) + \int_T \int_A w(|a - a'|) \mu_A(da'|t') \nu(dt'|t),$$

where $\mu_A(\cdot|t')$ is the distribution of actions of other players in the population, conditional on t' . Therefore, payoff of an agent in a single period is the sum of their idealistic utility and expected payoff to conformity relative to their interactions with other agents. Our specification implies that the social position cannot be contingent on the social statuses of other agents. It is chosen before any interaction occurs, highlighting the tension between idealism and conformism.

Following the rule *you become whom you pretend to be*, we assume that the social position in a current period has a direct impact on the identity in the following period. Formally, the transition is governed by cumulative probability distribution $q(t'|a)$ that determines the likelihood of the agent acquiring identity t' in the next period following their choice of a at the current date. We assume that function $a \rightarrow q(\cdot|a)$ is continuous and stochastically increasing in a .

The above game admits a greatest (and a least) MSNDE. Indeed, function r satisfies conditions (i), (ii), and (iv), (v) from Assumption 2. Moreover, since the transition kernel q depends only on a , it satisfies Assumption 3. Finally, as long as functions $\underline{a}, \bar{a}$ are continuous, in addition to the previously stated assumptions, correspondence $\tilde{A}(t, \tau) = [\underline{a}(t), \bar{a}(t)]$ is continuous, compact-valued, and satisfies strong complementarity. Finally, Assumption 1 holds as well.

Apart from equilibrium existence, one can determine equilibrium comparative transitional dynamics in the model. It is clear that, as the initial distribution of identities τ_1 shifts in the first-order stochastic sense, the equilibrium pair (μ^*, Φ^*) and the entire equilibrium transition path $\{\mu_n^*\}$ increase.

It is crucial that the transition function q depends *only* on action a . Following Remark 2, this allows us to dispense of the assumption that the function r and the correspondence $\tilde{A}$ are increasing in t , which is critical for this application.

4.3 Parenting and endogenous dynastic preferences

We can apply our tools to dynamic games with short-lived agents, where individuals make decisions in one period only, but their actions propel dynamics for future generations. This example is inspired by the literature on paternalistic bequests, keeping-up-with-the-Joneses, and growth with endogenous preferences.⁴⁰

Consider a society populated with a measure space of single-parent single-child families. Each individual (a parent) lives for a single period and a parent-child sequence forms a dynasty. The type of a parent is determined by their lifetime income $y \in [0, 1]$ and a parameter $i \in [0, 1]$ that summarizes preferences of the individual toward consumption, in a way that will be explained shortly. Therefore, in this setting, the types $t = (y, i)$ belong to $T = [0, 1]^2$.

⁴⁰See Cole, Mailath, and Postlewaite (1992), Doepke and Zilibotti (2017), and Genicot and Ray (2017).

Each period, the income can be devoted to consumption c and investment (savings) s , thus imposing the constraint $y = c + s$ for each dynasty. Consumption yields immediate utility $u(c, g)$, where the parameter g represents *propensity to consume*. We assume that the function u is continuous and concave in c , and has increasing differences in (c, g) . That is, the marginal utility of consumption for the current generation increases with g . Below, we specify how the variable g is related to the preference for consumption determined by the type i .

We assume paternalistic preferences, that is, the parent evaluates the well-being of their child with a function $w(t', \tau')$, where $t' = (y', i')$ is the future type of the child, and τ' is a distribution of types in the next period. We assume that w is increasing in $t = (y, i)$, thus the parent values high income and preference for consumption of the child. Since the parent cares only about their immediate descendant, they want the child to consume as much as possible. Moreover, let w have increasing differences in (t', τ') , that is, the higher is the future distribution of types, the higher is the parent's incremental benefit of the child's type.

Each parent devotes (e.g., educational) effort $e \in E = [0, 1]$ to shape preferences of their child (i.e., raise their aspiration level). The cost of effort is given by $C(e, \mu_E)$, where μ_E denotes the distribution of efforts in the population. We assume that the cost function is continuous and increasing with e , and has decreasing differences in (e, μ_E) —the higher effort in the entire population, the easier it is for an individual to influence their child.

Given our description, the action of an individual is $a = (s, e) \in A := [0, 1]^2$. Savings s and effort e affect both the future income and preferences of the child. Let the cumulative distribution $q(t'|a)$ determine the probability of the future type of the child being $t' = (y', i')$, where q is stochastically increasing in both arguments and is stochastically supermodular in $a = (s, e)$. Thus investment s and effort e are complements. Indeed, from the parent's perspective, higher effort (that skews preference of the child toward consumption) makes marginal investment/bequest more valuable. The more future income of the child is devoted to consumption, the more it pleases the paternalistic parent.

Finally, the marginal propensity to consume g is generated endogenously for each individual via keeping-up-with-the-Joneses effect. Formally, let $g = \theta\Gamma(t, \mu_C)$, for some positive parameter θ and an increasing function Γ that depends both on the type $t = (y, i)$ of the player and the distribution μ_C over the current consumption levels in the population. For example,

$$\Gamma(t, \mu_C) := \inf\{c \in [0, 1] : i \leq \mu_C(c'' \leq c)\}, \quad (10)$$

where $t = (y, i)$. That is, Γ is the i 'th quantile of consumption in the population.⁴¹

⁴¹This model is broadly related to Echenique and Komunjer (2009) and Doepke, Sorrenti, and Zilibotti (2019) concerning endogenous transmission of preferences in dynastic models of a household. Our model with quantile aspiration preferences and paternalism could be extended to altruistic dynastic choice, peer effects, or locational concerns as in Agostinelli, Doepke, Sorrenti, and Zilibotti (2020).

Given our description, the objective of a parent of type $t = (y, i)$ is to maximize

$$u(y - s, \theta \Gamma(t, \mu_C)) + \int_{[0,1]} w(t', \Phi_T(\mu)) q(dt'|s, e) - C(e, \mu_E),$$

with respect to $(s, e) \in \tilde{A}(t, \tau) = [0, y] \times [0, 1]$. Here, the mapping $\Phi_T(\mu)$ is the projected next-period distribution of types in the population. Notice that w is not a value function as discussed in Section 2; rather, it is a paternalistic evaluation of the child's welfare that may be misaligned with preferences of the child.

To verify whether assumptions of our theorems are satisfied, consider an increasing Markov strategy: $\sigma : T \rightarrow \mathcal{A}$, with σ_s and σ_e being its projections on both coordinates. Then we have $\mu_C(Z) = \tau(\{t \in T : [y - \sigma_s(t)] \in Z\})$, $\mu_E(Z) = \tau(\{t : \sigma_e(t) \in Z\})$, and $\Phi_T(\mu)(Z) = \int_T q(Z|\sigma_s(t), \sigma_e(t))\tau(dt)$, for some measurable set Z . Pointwise higher strategies σ imply a first-order stochastic dominance increase of μ_E and $\Phi_T(\mu)$, but a first-order stochastic dominance decrease in μ_C .⁴² Increasing differences of $u(c, g)$, $w(t', \tau')$, and $-C(e, \mu_E)$, together with assumptions on q suffice to show that it is optimal to play a strategy that is increasing in their private type. This suffices to show that there exist a greatest MSNDE (μ^*, Φ^*) , which can be approximated using successive iterations.

When considering ordered changes in the deep parameters of the model, we can apply our equilibrium comparative transitional dynamics and equilibrium approximation to these types of models. In particular, one can show the greatest (and the least) MSNDE is *decreasing* with respect to the parameter θ . These observations are true even though the payoff function is not necessarily increasing in t , nor it has increasing differences in (t, μ) . In fact, whenever function Γ is specified as in (10), the latter does not hold. Since the additional assumptions are required to show the particular properties of the value function in the infinite horizon problem, they play no role in dynamic games with short-lived agents.

4.4 Dynamic contests with coordination failures and learning

Consider a prototypical coordination game based on Angeletos and Lian (2016), with applications to beauty contests, bank runs, riot games, or currency attacks.⁴³ Here, we focus on a simple dynamic beauty contest. In this large dynamic game, each player receives a private signal t and chooses an action a every period. Action is costly and the cost depends on the type t , which is summarized in the utility function $u(t, a)$, where u is increasing in t and has increasing differences in (t, a) . In addition, the player's payoff depends on actions taken by other players, say $\int_{\mathcal{A}} g(a, \tilde{a}) \mu_{\mathcal{A}}(d\tilde{a})$, where the function g has increasing differences in $(a, \tilde{a})$.

As is standard in global games and dynamic coordination games with complementarities, we focus on monotone equilibria in which players use an increasing strategy

⁴²Indeed, $\int_C f(c) \mu'_C(dc) = \int_T f(y - \sigma'_s(t)) \tau(dt) \leq \int_T f(y - \sigma_s(t)) \tau(dt) = \int_C f(c) \mu_C(dc)$, for any measurable and increasing function $f : [0, 1] \rightarrow \mathbb{R}$, where σ'_s pointwise dominates σ_s .

⁴³See Morris and Shin (2002) for an extensive discussion of this literature. See also Carmona, Delarue, and Lacker (2017) for an application of mean-field methods to a related class of games.

$\sigma : T \rightarrow A$. The one-period payoff of an agent playing a is

$$r(t, a, \mu) := u(t, a) + \int_T g(a, \sigma(\tilde{t})) \mu_T(d\tilde{t}),$$

Such payoff satisfies assumptions required by Theorem 1, and so there exists a greatest MSNDE, where each player is using an increasing strategy σ .⁴⁴

The framework can be applied to *dynamic* riot games with private types, where

$$r(t, a, \mu) := a \left[\int_S (t_1 + L) \mathbf{1}_{\{R(\mu) \geq \tilde{s}\}} \nu(d\tilde{s}) - L \right] - c(a, t_2),$$

for some player type by $t = (t_1, t_2)$ and a compact interval $S \subseteq \mathbb{R}$. Thus, taking the risky action $a = 1$ allows the player to win t_1 if a sufficient number $R(\mu) := \mu(\{(t, a) : a = 1\})$ of players takes a risky (and costly) action, or lose L otherwise. The strength s of the police is distributed according to some measure ν . Whenever the cost function is decreasing in t_2 and $c(0, t_2) = 0$ (due to normalization), the dynamic game can be solved for a general transition functions $q(\cdot | t, a, \mu)$, thus allowing to model inertia, habit formation, or dynamic social externalities. See also Morris and Yildiz (2019) applications.

4.5 Idiosyncratic risk under multidimensional production externalities and technological dynamics

Our model can be applied to analyze dynamics of technological progress in large economies where agents face uninsurable private productivity risk. This includes the model of Romer (1986) in a Bewley–Huggett–Aiyagari type setting with ex-ante identical agents and ex post heterogeneity in production and no borrowing.⁴⁵

The economy is populated with a measure space of producers, each endowed with capital $t \in T = [0, 1]$, one unit of time, and a private technology f . The technology transforms private inputs into finished outputs. Moreover, its productivity depends on economy-wide externality summarized by the distribution of capital and labor in the economy. Specifically, each agent with t units of capital and expending $l \in L = [0, 1]$ units of time is able to produce $y = f(t, l, \mu_{T \times L})$ units of the finished output, where $\mu_{T \times L}$ is the distribution of capital-labor levels in the population. We assume that the production function f is continuous and increasing with respect to all arguments, and has increasing differences in (t, l) , (t, μ) , and (l, μ) .⁴⁶ In particular, the private technologies endowed to each agent need not be convex. In addition, our reduced form of technology allows for nontrivial interactions with market leaders, closely related companies, or a competitive fringe in both capital and labor dimensions.

The output can be devoted to consumption c or investment i ; hence, $c + i = y$. When c units of the output are consumed and labor supply is l , the agent receives utility

⁴⁴We may dispense monotonicity of u with respect to t as long as the transition function q depends only on the one-dimensional action a .

⁴⁵See also Angeletos and Calvet (2005) for a related study.

⁴⁶For example, function $f(t, l, \mu_{T \times L}) := \int_{T \times L} g(t, l, \tilde{t}, \tilde{l}) \mu_{T \times L}(d\tilde{t} \times d\tilde{l})$ would satisfy such conditions as long as g is supermodular in all arguments jointly.

$U(c, l) = u(c) + v(1 - l)$, where $u, v: \mathbb{R} \rightarrow \mathbb{R}$ are smooth, concave, and strictly increasing. Whenever $i \in I := [0, 1]$ units of the good is invested, the capital in the next period is determined stochastically with $q(\cdot|i)$.⁴⁷

To preserve the complementarity structure to the value functions, we require some known complementarity conditions for joint monotone controls (see [Hopenhayn and Prescott \(1992\)](#) and [Mirman, Morand, and Reffett \(2008\)](#)). Along those lines, we assume the standard condition $-u''/u' \leq f''_{12}/(f'_1 f'_2)$. It requires that degree of complementarity between private capital and labor is high relatively to the curvature of the utility function. This suffices for payoffs to have increasing differences in (t, l) . To guarantee increasing differences in $(t, \mu_{T \times L})$, we require that $u'(f(t, l, \mu_{T \times L}) - c)f'_1(t, l, \mu_{T \times L})$ is increasing in $\mu_{T \times L}$.⁴⁸ Analogous conditions guarantee increasing differences in $(l, \mu_{T \times L})$.⁴⁹

The above conditions are sufficient for Theorem 1 to hold. Therefore, there exist extremal MSNDEs for this large dynamic nonmarket economy (interpreted as a large anonymous game). Moreover, the extremal equilibria can be approximated using iterative methods. This example highlights the difference between our results and those in the existing literature. Specifically, we consider Markov stationary transitional dynamics and comparative dynamics results (in addition to comparative statics of the steady-state). For example, [Acemoglu and Jensen \(2015\)](#) discuss stochastic steady-state equilibria and the corresponding comparative statics given single dimensional aggregates that summarize production externalities.⁵⁰ Our conditions on the primitives, that guarantee that each player's value function has increasing differences in (t, μ) , are not crucial for their results.

4.6 OLG Bewley models

As assumed in Assumption 2 and pointed out in the discussion, our main result requires that the within-period payoff function r has increasing differences in the private type t and the measure μ . This condition implies that the value function v corresponding to the decision problem of each player has increasing differences in (t, μ) , and as a result, the equilibrium transition function Φ^* is increasing over its domain. Unfortunately,

⁴⁷Our methods allow to analyze *two sector economies*. A consumption good sector with technology f and investment good sector with stochastic technology $q(\cdot|i, l, \mu_{T \times L})$. In the example, we consider a simple version of q depending on investment i only.

⁴⁸Whenever the externality can be summarized with some increasing aggregate $G(\mu_{T \times L}) \in \mathbb{R}$, where $y = f(t, l, G(\mu_{T \times L}))$, the condition can be reduced to $-u''/u' \leq f''_{13}/(f'_1 f'_3)$.

⁴⁹Notice that, in our setting, the correspondence $A(t, l, \mu_{T \times L}) = [0, f(t, l, \mu_{T \times L})] \times L$ does not have strict complementarities. To assure that the value function v^* in (3) preserves increasing differences in (t, μ) , we need to use constructions of [Mirman, Morand, and Reffett \(2008\)](#) (Lemmas 11, 12 and Theorems 3 and 4). They state assumptions on u, v , and f under which the value function has increasing differences in (t, μ) . This example is useful to compare our results with the results of [Acemoglu and Jensen \(2015\)](#). We require increasing differences between controls and the aggregate distribution of assets to obtain the comparative statics of the extremal MSNDEs.

⁵⁰[Acemoglu and Jensen \(2015\)](#) identify positive shocks for a steady-state equilibrium. To preserve increasing differences between individual states and shock parameters, more assumptions are needed than those listed in their Lemma 1.

in general, the typical infinite horizon Bewley–Huggett–Aiyagari models violate this assumption.

For example, let the private type of each consumer in the economy be given by $t = (k, \ell)$, where k is the capital/asset endowment and ℓ is a random labor productivity draw. Each period, the agent can consume c or invest (save) $a \geq 0$ in the future capital, subject to the budget constraint $c + a = \rho(K)k + w(K)\ell$, where $\rho(K)$ and $w(K)$ denote the interest rate and the wage, respectively, that depend on the aggregate level of capital $K = \int k \mu_K(dk)$, where μ_K is the marginal distribution over private capital levels in the population. The within period utility of an individual is then given by

$$r(t, a, \mu) = u(\rho(K)k + w(K)\ell - a),$$

for some concave utility u . Whenever the price $\rho(K)$ is a decreasing function, the payoff r does not have increasing differences in (t, μ) , since $u(\rho(K)k + w(K)\ell - a)$ need not have increasing differences in $((k, \ell), K)$. Thus, we cannot assure that the value function corresponding to the dynamic problem has increasing differences in the investment a and the mean future capital K' . As a result, the equilibrium operator Φ^* need not be monotone and our results do not apply.

However, our methods can be applied to a version of the Diamond OLG model with idiosyncratic risk.⁵¹ Suppose that each period there is a measure space of representative young born, each endowed with a unit of free time and a (private) i.i.d. draw of labor productivity ℓ . Each old is endowed with a private (saved) capital k . The lifetime preferences for the young are

$$\frac{c_1^{1-\gamma}}{1-\rho} + \beta \frac{c_2^{1-\gamma}}{1-\gamma},$$

where $\gamma \geq 0$, and c_1, c_2 denote consumption of the consumer when young and old, respectively. Each young is supplying a unit of time inelastically and receives $w(K)$ wage per unit of efficiency. The problem of a young is then

$$\max_{a \in [0, w(K)\ell]} \frac{(w(K)\ell - a)^{1-\gamma}}{1-\gamma} + \beta \int \frac{(\rho(K')k)^{1-\gamma}}{1-\gamma} q(dk|a),$$

where K' is the future mean capital. Letting $\Phi_K(\mu)$ be the distribution of capital levels in the following period, as implied by transition function Φ , we have $K' = \int k d(\Phi_K(\mu))(k)$. The objective of each player has increasing differences in $(a, w(K)\ell)$ and (k, K') , for a decreasing $\rho(K)$ and $\gamma \geq 1$.⁵² Whenever the function q is increasing stochastically, the equilibrium transition map Φ is monotone, and thus the key condition required for our methods to work is satisfied.

⁵¹Here, we apply our methods to the study of short-lived players, as in Section 4.3.

⁵²Preferences need not be power utility for our methods to be applied. For example, if the lifetime preferences for the newly born young are given by $U(c_1, c_2) = u(c_1) + \beta v(c_2)$, where u continuous, concave, and v is $\mathcal{C}^2$, then $cv''/v' \geq 1$ will suffice. Critically, we require “income effect dominance” in interest rates, which is a type of “gross complements” condition, which induces increasing differences between individual investment choices and the distribution of assets.

APPENDIX

A.1 Mathematical glossary

A.1.1 Lattices and supermodularity

Posets, lattices, and chains A *partial order* $\geq_X$ over a set X is a reflexive, transitive, and antisymmetric binary relation. A partially ordered set, or a *poset*, is a pair $(X, \geq_X)$ consisting of a set X and a partial order $\geq_X$. When it causes no confusion, we denote $(X, \geq_X)$ with X .

For any $x, x' \in X$, their *infimum* (the greatest lower bound) is denoted by $x \wedge x'$, and their *supremum* (the least upper bound) by $x \vee x'$. The poset X is a *lattice* if for any $x, x' \in X$ both $x \wedge x'$ and $x \vee x'$ belong to X . Set A is a *sublattice* of X if $A \subseteq X$ and A is a lattice with the induced order, where $x \wedge x'$ and $x \vee x'$ are defined with $\geq_X$. A principle example of a lattice is the Euclidean space $(\mathbb{R}^\ell, \geq)$ endowed with the natural product order $\geq$, where $x' \geq x$ if $x'_i \geq x_i$, for all $i = 1, \dots, \ell$. In this case, $x \wedge x'$ and $x \vee x'$ are given by $(x \wedge x')_i = \min\{x_i, x'_i\}$ and $(x \vee x')_i = \max\{x_i, x'_i\}$, for all $i = 1, \dots, \ell$.

For any subset A of a poset X , we denote the *supremum* and *infimum* of A by $\vee A$ and $\wedge A$, respectively. That is, $\vee A$ is the least element in X such that $\vee A \geq a$, for all $a \in A$. Clearly, by definition, we have $x \vee x' = \vee\{x, x'\}$. We define $\wedge A$ analogously. A lattice X is *complete* if both $\vee A$ and $\wedge A$ belong to X for any $A \subseteq X$. We define a *complete sublattice* analogously.

A *chain* is a totally ordered poset, that is, all of its elements are ordered. A poset X is (countably) *lower chain complete* if any (countable) chain $A \subseteq X$ has its infimum in X . The poset is (countably) *upper chain complete* if any (countable) chain has its supremum in X . The poset is (countably) *chain complete* if it is both upper and lower (countably) chain complete.

Let $(\Delta, \geq)$ be a space of probability distributions over a compact subset of $S \subset \mathbb{R}^n$, endowed with the *first-order stochastic dominance ordering* $\geq$. That is, for any two probability measures $\mu, \nu \in \Delta$, we have $\mu \geq \nu$ in the first-order stochastic dominance sense, if $\int f(y)\mu(dy) \geq \int f(y)\nu(dy)$, for any measurable, bounded function $f : S \rightarrow \mathbb{R}$ that increases over S with respect to the corresponding ordering. In particular, $(\Delta, \geq)$ is a poset (see Kamae, Krengel, and O'Brien (1977)) but is *not* a lattice, unless S is a subset of $\mathbb{R}$. However, $(\Delta, \geq)$ is chain complete (see Lemma 2 in Balbus et al. (2015)).

Supermodularity Suppose that X is a lattice. A function $f : X \rightarrow \mathbb{R}$ is *supermodular* in x , if $f(x \wedge x') + f(x \vee x') \geq f(x) + f(x')$, for any $x, x' \in X$.

Suppose that X is a lattice and Δ is a space of probability distributions over a measurable poset S . The function $q : X \rightarrow \Delta$, taking values $q(\cdot|x) \in \Delta$, is *stochastically supermodular* in x if the function $f(x) := \int_S u(s)q(ds|x)$ is supermodular in x , for any measurable, bounded, and increasing function $u : S \rightarrow \mathbb{R}$.

Increasing differences For arbitrary posets X and T , function $f : X \times T \rightarrow \mathbb{R}$ has *increasing differences* in (x, t) if, for any $x' \geq_X x$ and $t' \geq_T t$, we have $f(x', t') - f(x, t') \geq f(x', t) - f(x, t)$.

The function $q : X \times T \rightarrow \Delta$ has *stochastically increasing differences* in (x, t) if $f(x, t) := \int_S u(s)q(ds|x, t)$ has increasing differences in (x, t) , for any measurable, bounded, and increasing function $u : S \rightarrow \mathbb{R}$.

Monotone correspondences Let X be a poset and Y be an arbitrary set. The correspondence $\Gamma : X \rightrightarrows Y$ is increasing with respect to *set inclusion* if $x' \geq_X x$ implies $\Gamma(x) \subseteq \Gamma(x')$.

For any poset X and a lattice Y , the correspondence $\Gamma : X \rightrightarrows Y$ is increasing with respect to the *strong set order* if, for any $x' \geq_X x$ and $y \in \Gamma(x)$, $y' \in \Gamma(x')$, we have $y \wedge y' \in \Gamma(x)$ and $y \vee y' \in \Gamma(x')$.

For any posets X , Z , and a lattice Y , the correspondence $\Gamma : X \times Z \rightrightarrows Y$ satisfies *strict complementarities* if for any $x' \geq_X x$, $z' \geq_Z z$, $y \in \Gamma(x, z)$, and $y' \in \Gamma(x', z)$, we have $y \wedge y' \in \Gamma(x, z)$ and $y \vee y' \in \Gamma(x', z')$. The following lemma is a generalization of Lemma 1 in [Hopenhayn and Prescott \(1992\)](#).

LEMMA A.1. *Take any posets X , Z and a lattice Y . Endow $X \times Z$ with a product order. If function $f : Y \times X \times Z \rightarrow \mathbb{R}$, taking values $f(y, x, z)$, is supermodular in y , has increasing differences in $(y, (x, z))$ and (x, z) , and the correspondence $\Gamma : X \times Z \rightrightarrows Y$ satisfies strict complementarities, then $g(x, z) := \sup_{y \in \Gamma(x, z)} f(y, x, z)$ has increasing differences in (x, z) .*

PROOF. Take any $x' \geq_X x$, $z' \geq_Z z$, and $y \in \Gamma(x, z')$, $y' \in \Gamma(x', z)$. Then

$$\begin{aligned} g(x, z) + g(x', z') &\geq f(y \wedge y', x, z) + f(y \vee y', x', z') \\ &= f(y \wedge y', x, z) - f(y \wedge y', x, z') + f(y \vee y', x', z') + f(y \wedge y', x, z') \\ &\geq f(y \wedge y', x, z) - f(y \wedge y', x, z') + f(y, x, z') + f(y', x', z'), \end{aligned}$$

where the first inequality follows from the definition of g and strict complementarity of Γ , and the second is implied by the fact that

$$\begin{aligned} &f(y \wedge y', x, z') + f(y \vee y', x', z') \\ &= [f(y \vee y', x', z') + f(y \wedge y', x', z')] - [f(y \wedge y', x', z') - f(y \wedge y', x, z')] \\ &\geq [f(y, x', z') + f(y', x', z')] - [f(y, x', z') - f(y, x, z')] \\ &= f(y, x, z') + f(y', x', z'), \end{aligned}$$

since f is supermodular in y (the first bracket) and has increasing differences in $(y, (x, z))$ (the second bracket). Thus

$$\begin{aligned} g(x, z) + g(x', z') &\geq f(y \wedge y', x, z) - f(y \wedge y', x, z') \\ &\quad + f(y, x, z') + f(y', x', z') + f(y', x', z) - f(y', x', z) \\ &= f(y, x, z') + f(y', x', z) + [f(y', x', z') - f(y \wedge y', x, z')] \\ &\quad - [f(y', x', z) - f(y \wedge y', x, z)] \geq f(y, x, z') + f(y', x', z), \end{aligned}$$

since f has increasing differences in $(y, (x, z))$ and (x, z) , thus in $((y, x), z)$.⁵³ We conclude by taking the supremum over the right-hand side of this inequality. $\square$

Monotone comparative statics Theorem 6.1 in [Topkis \(1978\)](#) states that for any lattice X and a poset Y , if the correspondence $\Gamma : Y \rightarrow X$ increases in the strong set order and a function $f : X \times Y \rightarrow \mathbb{R}$ is supermodular in y and has increasing differences in (x, y) , then the correspondence $x \rightrightarrows \arg \max_{y \in \Gamma(x)} f(x, y)$ is sublattice-valued and increasing in the strong set order. Moreover, if f is continuous in y and Γ is compact-valued, then the above correspondence is compact-valued and admits a greatest and a least selection, both increasing in x . See Theorem 6.2 in [Topkis \(1978\)](#).

Fixed points We present two theorems that are critical for proving Theorems 1 and 2. Given posets X and Y , a function $f : X \rightarrow Y$ is *increasing* if $x' \geq_X x$ implies $f(x') \geq_Y f(x)$. Below, we generalize Theorem 9 in [Markowsky \(1976\)](#).

PROPOSITION A.1. *Let $(X, \geq_X)$ be a lower chain complete poset with a greatest element. The set of fixed points of an increasing function $f : X \rightarrow X$ is a nonempty lower chain complete poset. Moreover, its greatest fixed point exists and is given by $\vee \{x \in X : f(x) \geq_X x\}$.⁵⁴*

PROOF. Let $\bar{x}$ be the greatest element of X . Let $\mathcal{J}$ be a set of ordinal numbers with cardinality strictly greater than X . Define the following transfinite sequence with the initial element $x_0 = \bar{x}$ and $x_i = \wedge \{f(x_j) : j < i\}$, for $i \in \mathcal{J} \setminus \{0\}$. We claim that $\{x_i\}$ is a well-defined decreasing sequence. Clearly, $x_1 = f(x_0) \leq x_0$. Suppose that $\{x_j\}_{j < i}$ is well-defined and decreasing for some i . Then $\{f(x_j)\}_{j < i}$ is a decreasing sequence, that has an infimum equal to x_i . Consequently, x_j is well-defined and decreasing on $[0, i]$. By transfinite induction, the transfinite sequence $\{x_i\}_{i \in \mathcal{J}}$ is well-defined and decreasing. Since $\mathcal{J}$ has the cardinality strictly greater than X , there is no one-to-one mapping between $\mathcal{J}$ and X . Consequently, take the least element $\bar{i}$ in $\{i \in \mathcal{J} : x_i = x_{i+1}\}$. Then $x_{\bar{i}} = x_{\bar{i}+1} = f(x_{\bar{i}})$, and $e^* := x_{\bar{i}}$ is a fixed point of f . To show that $e^* = \vee \{x \in X : f(x) \geq x\}$, set $\mathcal{X} := \{x \in X : f(x) \geq x\}$. Obviously, we have $e^* \in \mathcal{X}$. For any other $y \in \mathcal{X}$, we have $y \leq x_0$. Suppose there is $i \in \mathcal{J}$ such that $y \leq x_j$, for any $j < i$. Since $y \in \mathcal{X}$, by transfinite induction, we have $y \leq f(y) \leq f(x_j)$. Thus $y \leq \wedge \{f(x_j) : j \leq i\}$ and $y \leq x_i$, for any $i \in \mathcal{J}$, including $\bar{i}$. $\square$

Given posets X and Y , function $f : X \rightarrow Y$ is *monotone sup-preserving* if, for any increasing sequence $\{x_k\}_{k \in \mathbb{N}}$, we have $f(\vee \{x_k\}_{k \in \mathbb{N}}) = \vee \{f(x_k)\}_{k \in \mathbb{N}}$. It is *monotone inf-preserving* if $f(\wedge \{x_k\}_{k \in \mathbb{N}}) = \wedge \{f(x_k)\}_{k \in \mathbb{N}}$, for any decreasing sequence $\{x_k\}_{k \in \mathbb{N}}$. Below

⁵³Indeed, take any $y' \geq_Y y$, $x' \geq_X x$, and $z' \geq_Z z$. If f has increasing differences in $(y, (x, z))$, then $f(y', x', z') - f(y, x', z') \geq f(y', x', z) - f(y, x', z)$, which is equivalent to $f(y', x', z') - f(y', x', z) \geq f(y, x', z') - f(y, x', z)$. Given that f has increasing differences in (x, z) , we have $f(y, x', z') - f(y, x', z) \geq f(y, x, z') - f(y, x, z)$. Combining the two inequalities yields $f(y', x', z') - f(y, x, z') \geq f(y', x', z) - f(y, x, z)$.

⁵⁴There is an obvious order dual to this result for increasing functions defined on an upper chain complete domain X that implies existence of the least fixed-point theorem.

we extend the classic comparative statics results of [Topkis \(1998\)](#) to countably chain complete posets. The result is an extension of the Tarski–Kantorovich theorem. See [Balbus, Reffett, and Woźny \(2015b\)](#).⁵⁵

PROPOSITION A.2. *Let X be a lower countably chain complete poset with the greatest element $\vee X$, and let Θ be a poset. For any function $f : X \times \Theta \rightarrow X$ and $\theta \in \Theta$ such that $f(\cdot, \theta)$ is increasing and monotone inf-preserving over X , the greatest fixed point of $f(\cdot, \theta)$ is given by $\bigwedge \{f(\vee X, \theta)\}_{n \in \mathbb{N}}$.⁵⁶ In addition, if f is increasing in the product order and $f(\cdot, \theta)$ is monotone inf-preserving, for all $\theta \in \Theta$, then the greatest fixed point is increasing over Θ .*

We now discuss relations between order convergence in $\mathcal{M}$ and weak convergence of measures.

REMARK A.1. As in the main section, let $\mathcal{M}$ be a set of probability measures over a compact subset of $\mathbb{R}^n$ endowed with the first order stochastic dominance relation $\succeq$. Since we operate on sequences that are either increasing or decreasing, the supremum or infimum in $\mathcal{M}$ is the limit not only in the (interval) topology generated by open intervals, but also in the weak topology on $\mathcal{M}$ (see [Hopenhayn and Prescott \(1992\)](#), pp. 1389–1391). That is, if $\{\mu_n\}$ is a first order stochastically increasing sequence with the supremum μ , then

$$\lim_{n \rightarrow \infty} \mu_n = \mu = \bigvee_{n=1}^{\infty} \mu_n,$$

where the limit on the left is defined in the sense of weak convergence on $\mathcal{M}$, and the supremum on the right is defined with respect to $(\mathcal{M}, \succeq)$. The same applies to the mappings $\Phi \in \mathcal{D}$. Recall that $\mathcal{D}$ includes inf-preserving mappings from $\mathcal{M}$ to itself. In fact, a weakly continuous self-map on $\mathcal{M}$ is monotone sup-inf preserving, but the converse does not hold.

A.1.2 The exact law of large numbers

Superatomless probability space Let $(\Lambda, \mathcal{L}, \lambda)$ be a probability space. For any $E \in \mathcal{L}$ such that $\lambda(E) > 0$, let $\mathcal{L}^E := \{E \cap E' : E' \in \mathcal{L}\}$ and λ^E be the re-scaled measure from the restriction of λ to $\mathcal{L}^E$. Let $\mathcal{L}_\lambda^E$ be the set of equivalence classes of sets in $\mathcal{L}^E$ such that $\lambda^E(E_1 \Delta E_2) = 0$, for $E_1, E_2 \in \mathcal{L}^E$.⁵⁷ We endow the space with metric $d^E : \mathcal{L}_\lambda^E \times \mathcal{L}_\lambda^E \rightarrow \mathbb{R}$ given by $d^E(E_1, E_2) := \lambda^E(E_1 \Delta E_2)$.

DEFINITION A.1 (Superatomless space). A probability space $(\Lambda, \mathcal{L}, \lambda)$ is superatomless if for any $E \in \mathcal{L}$ with $\lambda(E) > 0$, the space $(\mathcal{L}_\lambda^E, d^E)$ is nonseparable.⁵⁸

⁵⁵There is a dual version of this theorem for the least fixed point of the monotone sup-preserving function defined over an upper countably chain complete domain X .

⁵⁶By f^n we denote the n 'th composition of f , that is, $f^n = f \circ f \circ \dots \circ f$ (n times).

⁵⁷We denote $E_1 \Delta E_2 := (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$.

⁵⁸This definition is by [Podczeck \(2009, 2010\)](#). Equivalently, [Hoover and Keisler \(1984\)](#) and [Keisler and Yeneng \(2009\)](#) dub such spaces $\aleph_1$ -atomless and rich, respectively.

As shown by Podczeck (2009), any atomless Borel probability measure on a Polish space can be extended to a superatomless probability measure.

Fubini extension For any set Ω and $E \subseteq (\Lambda \times \Omega)$, we denote its sections by $E_\alpha := \{\omega \in \Omega : (\alpha, \omega) \in E\}$ and $E_\omega := \{\alpha \in \Lambda : (\alpha, \omega) \in E\}$, for any $\alpha \in \Lambda$ and $\omega \in \Omega$. Similarly, for any function f defined over $\Lambda \times \Omega$, let f_α and f_ω denote the section of f for a fixed α, ω , respectively. Consider the following definition.

DEFINITION A.2 (Fubini extension). The probability space $(\Lambda \times \Omega, \mathcal{L} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a *Fubini extension* of the natural product of probability spaces $(\Lambda, \mathcal{L}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ if: (i) $\mathcal{L} \boxtimes \mathcal{F}$ includes all sets from $\mathcal{L} \otimes \mathcal{F}$ and (ii) for an arbitrary set $E \in \mathcal{L} \boxtimes \mathcal{F}$ and $(\lambda \otimes P)$ -almost every $(\alpha, \omega) \in \Lambda \times \Omega$, the sections E_α and E_ω are $\mathcal{F}$ - and $\mathcal{L}$ -measurable, respectively, while

$$(\lambda \boxtimes P)(E) = \int_{\Lambda} \lambda(E_\omega) P(d\omega) = \int_{\Omega} P(E_\alpha) \lambda(d\alpha).$$

Given a probability space $(\Lambda, \mathcal{L}, \lambda)$, a collection of random variables $(X_\alpha)_{\alpha \in \Lambda}$ is *essentially pairwise independent*, if for $(\lambda \otimes \lambda)$ -almost every $(\alpha, \alpha') \in \Lambda \times \Lambda$, random variables X_α and $X_{\alpha'}$ are independent. A Fubini extension is *rich*, if there is a $(\mathcal{L} \boxtimes \mathcal{F})$ -measurable function $X : \Lambda \times \Omega \rightarrow \mathbb{R}$ such that the random variables $(X_\alpha)_{\alpha \in \Lambda}$ are essentially pairwise independent and the random variable X_α has the uniform distribution over $[0, 1]$, for λ -almost every $\alpha \in \Lambda$. Podczeck (2010) shows that there exists a rich Fubini extension if and only if the space is superatomless. Moreover, without loss, one may assume the random variables $(X_\alpha)_{\alpha \in \Lambda}$ to be independent, rather than pairwise-independent.

The (exact) law of large numbers A process is a $(\mathcal{L} \boxtimes \mathcal{F})$ -measurable function with values in a Polish space. For any process f and set $E \in \mathcal{L}$ such that $\lambda(E) > 0$, we denote the restriction of f to $E \times \Omega$ by f^E . Naturally, $\mathcal{L}^E \boxtimes \mathcal{F} := \{W \in \mathcal{L} \boxtimes \mathcal{F} : W \subseteq E \times \Omega\}$ and $(\lambda^E \boxtimes P)$ is a probability measure rescaled from the restriction of $(\lambda \boxtimes P)$ to $(\mathcal{L}^E \boxtimes \mathcal{F})$. The following is due to Sun (2006).

PROPOSITION A.3 (Law of large numbers). Let f be a process from a rich Fubini extension $(\Lambda \times \Omega, \mathcal{L} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to some Polish space. Then, for all $E \in \mathcal{L}$ such that $\lambda(E) > 0$ and P -almost every $\omega \in \Omega$, we have $\lambda(f_\omega^E)^{-1} = (\lambda^E \boxtimes P)(f^E)^{-1}$.

An iterative application of this proposition and universality of the rich Fubini extension imply the exact law of large numbers for the dynamic transition in equation (1).⁵⁹ Indeed, given any distribution μ_0 , we can define (μ_n) recursively,

$$\mu_{t+1} = \int_{T \times A} q(t, a, \mu_n) d\mu_n$$

Then there exists an implied probability measure μ^∞ on $(T \times A)^\infty$. Any random variable h taking values in $(T \times A)^\infty$ with distribution μ^∞ is a Markov process with state space

⁵⁹We thank the anonymous referee for suggesting this proof of the dynamic version of ELLN.

$T \times \mathcal{A}$, initial distribution μ_0 and transition probability $q(t, a, \mu_n)$ from time n to time $n + 1$. Sun's result on the universality of a rich Fubini extension on $\Lambda \times \Omega$ says that there is an essentially pairwise independent process F from $\Lambda \times \Omega$ to $(T \times \mathcal{A})^\infty$ such that for each $\alpha \in \Lambda$, F_α has the same distribution μ^∞ . Then Sun's ELLN says that, for a.e. ω , F_ω has the same distribution μ^∞ , which means that the type-action distribution in the n th period for all the agents is μ_n . See a related construction derived for a class of Bewley models by Cao (2020).

A.2 Proofs

We begin with a few ancillary results.

LEMMA A.2. *Let $(\Xi, \geq)$ be a poset with its order topology, and $\{f_k\}$ be a sequence of increasing and monotone inf-preserving functions $f_k : \Xi \rightarrow \mathbb{R}$. Whenever $x_k \downarrow x$ in Ξ and $f_k \downarrow f$ (pointwise), then $f_k(x_k) \rightarrow f(x)$.*

PROOF. Let $n \in \mathbb{N}$. Since $\{f_k\}$ is a decreasing sequence of increasing functions and $x_k \downarrow x$, then $k \geq n$ implies $f(x) \leq f_k(x_k) \leq f_k(x_n)$. Thus we have $f(x) \leq \liminf_{k \rightarrow \infty} f_k(x_k) \leq \limsup_{k \rightarrow \infty} f_k(x_k) \leq f(x_n)$. To conclude, let $n \rightarrow \infty$. $\square$

LEMMA A.3. *Let $\{\nu_k\}$ be a sequence of probability measures on a Polish space S , and $\{h_k\}$ be a sequence of bounded, measurable functions $h_k : S \rightarrow \mathbb{R}$. If $\nu_k \downarrow \nu$ (stochastically and in weak topology) and $h_k \downarrow h$, then $\lim_{k \rightarrow \infty} \int h_k d\nu_k = \int h d\nu$.*

PROOF. It is a consequence of Lemma A.2, where Ξ is a space of bounded, measurable, real valued functions on S , and $f_k(x) := \int_S x(s) \nu_k(ds)$, $x_k(s) = h_k(s)$. $\square$

LEMMA A.4. *Let S_1, S_2 be topological spaces and $f : S_1 \times S_2 \rightarrow \mathbb{R}$ be a continuous function. Let $\Gamma : S_1 \rightrightarrows S_2$ be a continuous and compact-valued correspondence, and define $\Gamma^*(x) := \arg\max_{y \in \Gamma(x)} f(x, y)$. If $x_k \rightarrow x$ in S_1 , $y_k \rightarrow y$ in S_2 , and $y_k \in \Gamma^*(x_k)$, then $y \in \Gamma^*(x)$.*

PROOF. Let $y' \in \Gamma(x)$. By continuity of Γ , for any $k \in \mathbb{N}$, there is $y'_k \in \Gamma(x_k)$ such that $y'_k \rightarrow y'$. Since $y_k \in \Gamma^*(x_k)$, we have $f(x_k, y_k) \geq f(x_k, y'_k)$, for all $k \in \mathbb{N}$. By continuity of f , this implies that $f(x, y) \geq f(x, y')$. Since $y' \in \Gamma(x)$ is arbitrary, we have $y \in \Gamma^*(x)$. $\square$

We proceed with the proofs that were omitted in the main paper.

Proof of Lemma 1 Consider $v_n \in \mathcal{V}$, for all $n \in \mathbb{N}$, and $v_n \rightarrow v$. Let (μ_k) and (Φ_k) be a collection of decreasing sequences in $\mathcal{M}$ and $\mathcal{D}$, respectively, such that $\mu_k \rightarrow \mu$ (weakly) and $\Phi_k \rightarrow \Phi$ (pointwise). Take any $t \in T$ and $\epsilon > 0$. There is $n_0 \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$ and $n \geq n_0$, we have

$$\begin{aligned} & |v(t, \mu_k, \Phi_k) - v(t, \mu, \Phi)| \\ & \leq |v(t, \mu_k, \Phi_k) - v_n(t, \mu_k, \Phi_k)| + |v_n(t, \mu_k, \Phi_k) - v_n(t, \mu, \Phi)| \\ & \quad + |v_n(t, \mu, \Phi) - v(t, \mu, \Phi)| \leq \frac{2}{3}\epsilon + |v_n(t, \mu_k, \Phi_k) - v_n(t, \mu, \Phi)| \end{aligned} \quad (11)$$

Take any $n \in \mathbb{N}$ satisfying (11). Since $v_n \in \mathcal{V}$, for large enough k , we have $|v_n(t, \mu_k, \Phi_n) - v_n(t, \mu, \Phi)| \leq \epsilon/3$. Given (11), $|v(t, \mu_k, \Phi_k) - v(t, \mu, \Phi)| < \epsilon$, for a large k . Hence, v is monotone inf-preserving. Thus $v \in \mathcal{V}$. $\square$

Continuation of the proof to Lemma 4 We prove (vi). Using Assumption 2, definition of $\mathcal{V}$, and Lemma A.4, one can show that F is a Carathéodory function in (t, a) , that is, measurable in t and continuous in a . Hence, by Assumption 1 and measurable maximum theorem (Theorem 18.19 in Aliprantis and Border (2006)) the correspondence $\Gamma(t, \mu; v, \Phi)$ is measurable in t ; hence, weakly measurable.⁶⁰ For each $j = 1, 2, \dots, k$, the function $\pi_j(t) := \max_{a \in \Gamma(t, \mu; v, \Phi)} a_j$ is measurable (again by the measurable maximum theorem). Thus $t \rightarrow \bar{\gamma}(t, \mu, \Phi; v) = (\pi_1(t), \pi_2(t), \dots, \pi_k(t))$ is measurable. $\square$

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⁶⁰See, for example, Lemma 18.2 in Aliprantis and Border (2006).

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