

On the revealed preference analysis of stable aggregate matchings

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Echenique, Lee, Shum, and Yenmez (2013) established the testable revealed preference restrictions for stable aggregate matching with transferable and nontransferable utility and for extremal stable matchings. In this paper, we rephrase their restrictions in terms of properties on a corresponding bipartite graph. From this, we obtain a simple condition that verifies whether a given aggregate matching is rationalizable. For matchings that are not rationalizable, we provide a simple greedy algorithm that computes the minimum number of matches that need to be removed to obtain a rationalizable matching. We also show that the related problem of finding the minimum number of types that we need to remove in order to obtain a rationalizable matching is NP-complete.

KEYWORDS. Revealed preference theory, two-sided matching markets, stability, computational complexity, matroid.

JEL CLASSIFICATION. C78, D11.

1. INTRODUCTION

This paper revisits the results of Echenique et al. (2013) (ELSY from now on) by investigating the testable revealed preference implications of stable aggregate matchings in a transferable utility (TU) and nontransferable utility (NTU) setting.¹ For their characterization, Echenique et al. (2013) start by mapping a given aggregate matching to a particular graph. They show that the matching is NTU rationalizable if and only if this graph has no two distinct, regular, and vertex-minimal cycles that are connected.² Subsequently, they show that an aggregate matching is TU rationalizable or NTU rationalizable by an extremal stable matching if and only if the associated graph has no regular vertex-minimal cycle.

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¹See also Echenique (2008), Cherchye, Demuynck, De Rock, and Vermeulen (2017), and Hu, Li, and Tang (2020) for other revealed preference approaches to matching models.

²See the next section for a formal definition of the various concepts.

The contribution of this paper is threefold. First, we derive an easy equivalent set of conditions to verify these rationalizability conditions. Our conditions only involve comparing the number of vertices and edges on the connected components of a corresponding bipartite graph and are therefore very easy to verify. Next, we look at matchings where the rationalizability conditions are not satisfied. Relying on results from matroid theory, we provide efficient greedy algorithms to determine the minimum number of matches that need to be removed in order to obtain a matching that is rationalizable. This provides us with a goodness-of-fit measure to determine how close a given matching is to being rationalizable. Another way to restore rationalizability is by removing types. This begs the question of determining the minimal number of types that must be removed in order to obtain a matching that is rationalizable. We show that this problem is NP-hard.

The paper unfolds as follows. Section 2 sets the stage by describing the framework and ELSY's characterization. In Section 3, we present an equivalent characterization in terms of a bipartite graph. Section 4 looks at the problem of removing a minimal number of matches to obtain a matching that is rationalizable. In turn, Section 5 looks at the problem of removing a minimal number of types in order to obtain a rationalizable matching. Section 6 contains a short empirical illustration. Section 7 contains a conclusion. All proofs are in the Appendix.

2. FRAMEWORK AND CONTRIBUTION

In this section, we introduce the notation and definitions necessary to state the main result of ELSY.

There are two disjoint finite sets of types, denoted by M and W . The set M is the set of types of men and W is the set of types of women. For each type combination $(m, w) \in M \times W$, we know the number of matchings (marriages) that take place between these types. We denote this number by $X(m, w) \in \mathbb{N} \cup \{0\}$.³ We define an (aggregate) matching by a triple $\mathcal{M} = (M, W, X)$.

Similar to ELSY, we ignore singles. In addition, we will also abstain from including individual rationality constraints as a stability requirement. For many data sets on matchings, singles are not observed or are assumed to have preferences different than matched individuals. If so, omitting them from the analysis is without loss of generality. On the other hand, the model can easily be extended to include singles by adding on each side of the market an additional type representing “singlehood.” See, for example, the working paper of Echenique, Lee, and Shum (2010) on how to incorporate singles into the current framework.

The nontransferable utility setting Following the revealed preference methodology, the goal is to assign preferences for each type of men $m \in M$ over all types of women $w \in W$ and preferences for all types of women $w \in W$ over all types of men $m \in M$, such that the observed matching is stable. Let $P_M = (P_m)_{m \in M}$ be a preference profile for the types of

³It is easy to generalize the setting to $X(m, w) \in \mathbb{R}_+$, for example, to account for random matchings.

men in M and let $P_W = (P_w)_{w \in W}$ be a preference profile for the types of women in W . Every preference P_m and P_w is a strict linear order over W and M , respectively.

The pair $(m, w) \in M \times W$ is a *blocking pair* if there exists a type $m' \in M$ and a type $w' \in W$ such that

$$m P_w m', \quad w P_m w', \quad \text{and both} \quad X(m', w) > 0 \quad \text{and} \quad X(m, w') > 0.$$

This means that type w women prefer men of type m over men of type m' and that type m men prefer women of type w over women of type w' , although there are type m men matched to type w' women and type w women matched to men of type m' . The matching $\mathcal{M} = (M, W, X)$ is *stable* for the preferences (P_M, P_W) if there are no blocking pairs.

Although in reality we may observe the matching $\mathcal{M} = (M, W, X)$, we usually do not observe the preference profiles P_M and P_W . If it is possible to find preference profiles P_M and P_W such that the observed matching is stable, then the matching is said to be *NTU rationalizable*.

DEFINITION (NTU Rationalizability). A matching $\mathcal{M} = (M, W, X)$ is NTU rationalizable if and only if there exists a preference profile P_M and a preference profile P_W such that $\mathcal{M}$ is stable for the preferences (P_M, P_W) .

An extremal stable matchings is a matching that is better for one side of the market (and worse for the other side) than any other stable matching. Here, better is in the sense of first-order stochastic dominance.⁴ Men optimal stable matchings are called M-optimal while women optimal matchings are W-optimal.

DEFINITION (M- (W-) Extremal Rationalizability). A matching $\mathcal{M} = (M, W, X)$ is M- (W-) rationalizable if and only if there exists a preference profile P_M and a preference profile P_W such that $\mathcal{M}$ is the M- or W-optimal stable matching for the preferences (P_M, P_W) .

The transferable utility setting In the TU setting, every type-man–type-woman combination obtains a surplus $\alpha(m, w)$ that can be divided between the two partners. In such TU settings, stable matchings are those that maximize joint surplus (Shapley and Shubik (1972)):

$$\begin{aligned} \max \quad & \sum_{Z(m,w) \in \mathbb{N} \cup \{0\}} \alpha(m, w) Z(m, w), \\ \text{such that} \quad & \sum_{m \in M} Z(m, w) = \sum_{m \in M} X(m, w) \quad \forall w \in W \\ & \sum_{w \in W} Z(m, w) = \sum_{w \in W} X(m, w) \quad \forall m \in M. \end{aligned} \tag{OP1}$$

⁴See Echenique et al. (2013) for the exact definition.

The two restrictions in the program are adding up constraints and imposing that the matching (M, W, Z) should have, for each man type and woman type, the same number of individuals as in the observed matching.

The matching (M, W, Z) is TU stable if Z is a solution of (OP1). In reality, however, we do not observe the surplus values $\alpha(m, w)$. Following the revealed preference methodology, the aim is to find values $\alpha(m, w)$ such that the observed matching $\mathcal{M} = (M, W, X)$ solves (OP1).

DEFINITION (TU Rationalizability). A matching $\mathcal{M} = (M, W, X)$ is TU rationalizable if and only if there exist surplus values $(\alpha(m, w))_{m \in M, w \in W}$ such that $\mathcal{M} = (M, W, X)$ is TU stable, i.e., solves (OP1).

Graph theoretic concepts In order to state the rationalizability characterizations of ELSY, we need to introduce some graph theoretic notation and definitions.

An undirected *graph* or network $G = (V, E)$ consists of a finite set of *vertices* or nodes, V , and *edges*, E , where each edge $e \in E$ is a two element subset of V . We usually write edges as $e = (x, y)$ instead of $e = \{x, y\}$, but it should be understood that (x, y) and (y, x) represent the same edge.

A *path* in a graph $G = (V, E)$ is a sequence of distinct vertices in V , say $\rho = \langle x_0, x_1, \dots, x_n \rangle$ such that $(x_i, x_{i+1}) \in E$ for all $i = 0, \dots, n - 1$. Two vertices $x, y \in V$ are *connected* if there is a path from x to y .

A *cycle* in a graph $G = (V, E)$ is a sequence of vertices $\gamma = \langle x_0, x_1, x_2, \dots, x_n, x_0 \rangle$ such that (i) for all $i = 1, \dots, n - 1$, $(x_i, x_{i+1}) \in E$ and $(x_n, x_0) \in E$, (ii) all these edges are distinct, and (iii) all vertices $x_0, \dots, x_n$ are also distinct.

A cycle $\gamma = \langle x_0, x_1, \dots, x_n, x_0 \rangle$ is *vertex-minimal* if there is no proper subset of vertices that also form a cycle. As an illustration, in Figure 1, the cycle $\gamma = \langle a, b, c, d, e, a \rangle$ is not vertex-minimal as $\gamma' = \langle a, b, d, e, a \rangle$ is a cycle with fewer vertices.

If γ and γ' are two cycles and there is a path from a vertex in γ to a vertex in γ' in G , we say that the two cycles are connected.

The graph $\tilde{G} = (\tilde{V}, \tilde{E})$ is a *subgraph* of $G = (V, E)$ if $\tilde{V} \subseteq V$ and $\tilde{E}$ contains all edges from E that have both vertices in $\tilde{V}$:

$$\tilde{E} = \{(x, y) \in E : x, y \in \tilde{V}\}.$$

A *connected component* of $G = (V, E)$ is a subgraph $\tilde{G} = (\tilde{V}, \tilde{E})$ such that any two vertices in $\tilde{V}$ are connected via some path in $\tilde{G}$ and no vertex in $\tilde{V}$ is connected to a vertex

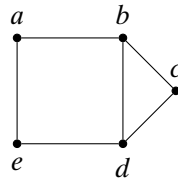
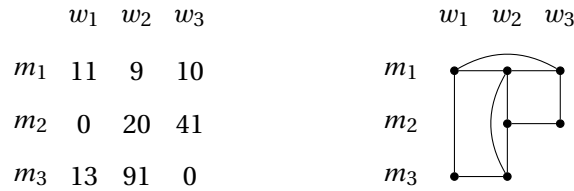


FIGURE 1. The cycle $\langle a, b, c, d, e \rangle$ is not vertex-minimal.

FIGURE 2. A matching and its graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$.

in $V \setminus \tilde{V}$ via some path in G . Every graph can be partitioned into its connected components. Two vertices are in the same connected component of G if and only if they are connected in G .

A *tree* is a connected graph with no cycles. If every connected component of a graph is a tree, the graph is called a *forest*.

Generalizing this notion, we call a connected graph a *pseudotree* if it contains at most one cycle. If every connected component of a graph is a pseudotree, the graph is called a *pseudoforest*.

The graph $\mathbb{G}$ Starting from a matching $\mathcal{M} = (M, W, X)$, ELSY define a particular graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, where $\mathbb{V}$ consists of all type combinations (m, w) that have at least one matching:

$$\mathbb{V} = \{(m, w) \in M \times W : X(m, w) > 0\}.$$

Next, the graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ has an edge between a vertex (m, w) and a vertex (m', w') if and only if $m = m'$ or $w = w'$:

$$\mathbb{E} = \{((m, w), (m', w')) \in \mathbb{V}^2 : m = m' \text{ or } w = w'\}.$$

Figure 2 gives an example reproduced from ELSY. The left panel shows a matching $\mathcal{M} = (M, W, X)$ for three types of men and three types of women. The value at row m_i and column w_j is given by $X(m_i, w_j)$. The panel on the right gives the graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$. There is a node for every nonzero entry in the matching table and an edge between every two nodes in a common row or column.

We call a cycle γ in $\mathbb{G}$ regular if it involves at least two types of men and two types of women. For example, in Figure 2 the following cycle is regular:

$$\gamma = \langle (m_1, w_1), (m_1, w_2), (m_3, w_2), (m_3, w_1), (m_1, w_1) \rangle.$$

On the other hand, the the following cycle is not regular:

$$\gamma' = \langle (m_1, w_2), (m_3, w_2), (m_2, w_2), (m_1, w_2) \rangle.$$

The following theorem gives the characterization of ELSY.

THEOREM 1 (Echenique et al. (2013)). (i) A matching $\mathcal{M} = (M, W, X)$ is NTU rationalizable if and only if the graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ does not contain two distinct, regular, vertex-minimal cycles that are connected.

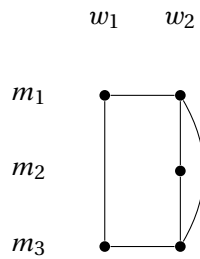


FIGURE 3. Two connected cycles that are not vertex-minimal.

- (ii) A matching $\mathcal{M} = (M, W, X)$ is TU rationalizable, M -extremal rationalizable, or W -extremal rationalizable if and only if the graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ does not contain a regular, vertex-minimal cycle.

As the revealed preference conditions for TU rationalizability and M - (W -) extremal rationalizability are equivalent, we will refer to the two cases simply as TU rationalizability from now on.

The example in Figure 2 is not NTU (TU) rationalizable. Indeed, there are two regular, vertex-minimal cycles that are connected:

$$\gamma_1 = \langle (m_1, w_1), (m_1, w_2), (m_3, w_2), (m_3, w_1), (m_1, w_1) \rangle$$

$$\gamma_2 = \langle (m_1, w_2), (m_1, w_3), (m_2, w_3), (m_2, w_2), (m_1, w_2) \rangle.$$

Figure 3 illustrates why we need to requirement vertex-minimality of the cycles. The graph has two connected regular cycles

$$\gamma_1 = \langle (m_1, w_1), (m_3, w_1), (m_3, w_2), (m_1, w_2), (m_1, w_1) \rangle$$

and

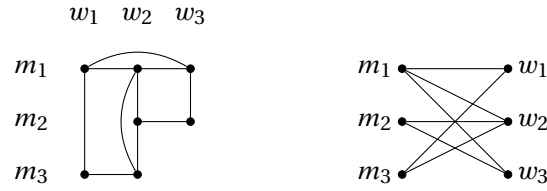
$$\gamma_2 = \langle (m_1, w_1), (m_3, w_1), (m_3, w_2), (m_2, w_2), (m_1, w_2), (m_1, w_1) \rangle.$$

The cycle γ_2 , however, is not vertex-minimal as the node (m_2, w_2) can be removed to obtain the cycle γ_1 . As such, the graph is still NTU rationalizable although it has two connected regular cycles.

If we want to verify whether a given matching is rationalizable, we could proceed by directly verifying the condition in Theorem 1. For example, for NTU rationalizability, one could try to enumerate all regular vertex-minimal cycles in the network $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ and check whether two of them are connected. Doing so, however, is not straightforward. As we will show in Theorem 2 below, there is a much more efficient way to verify rationalizability. In order to do so, we need to introduce an alternative representation of the matching in terms of a bipartite graph.

3. THE BIPARTITE GRAPH

Our results rely on the construction of a bipartite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ from the matching $\mathcal{M} = (M, W, X)$. We will use the calligraphic notation $\mathcal{G}$, $\mathcal{V}$, and $\mathcal{E}$ to denote this graph.

FIGURE 4. The graphs $\mathbb{G}$ (left) and $\mathcal{G}$ (right).

The vertices $\mathcal{V}$ are given by the set of types

$$\mathcal{V} = M \cup W.$$

Next, we have an edge between a vertex m and a vertex w if and only if they have some matches:

$$\mathcal{E} = \{(m, w) \in M \times W : X(m, w) > 0\}.$$

Figure 4 gives an illustration. The left part reproduces $\mathbb{G}$ from Figure 2. The right graph gives the corresponding bipartite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Vertices in $\mathbb{G}$ are related to edges in $\mathcal{G}$, while edges in $\mathbb{G}$ correspond to a pair of connected edges in $\mathcal{G}$.

The following result shows how rationalizability can alternatively be verified using the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

THEOREM 2. (i) *The following conditions are equivalent:*

- (i.a) *The matching $\mathcal{M} = (M, W, X)$ is NTU rationalizable.*
- (i.b) *The graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ has no two distinct, regular, vertex-minimal cycles that are connected.*
- (i.c) *The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has no two connected cycles.*
- (i.d) *Every connected component of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has at least as many vertices as edges.*

(ii) *The following conditions are equivalent:*

- (ii.a) *The matching $\mathcal{M} = (M, W, X)$ is TU rationalizable.*
- (ii.b) *The graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ has no regular, vertex-minimal cycle.*
- (ii.c) *The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has no cycle.*
- (ii.d) *Every connected component of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has more vertices than edges.*

The equivalences (i.a)–(i.b) and (ii.a)–(ii.b) repeat the characterization of ELSY from Theorem 1. Our main contribution is the equivalence (i.b)–(i.c) and (ii.b)–(ii.c). These show the connection between regular, vertex-minimal cycles of $\mathbb{G}$ and cycles in the bipartite graph $\mathcal{G}$. Notice that for the cycles in $\mathcal{G}$ we do not impose any requirement in

terms of vertex-minimality. Also, as every cycle in $\mathcal{G}$ contains at least two man types and two woman types (due to the “bipartiteness” of $\mathcal{G}$), every cycle in $\mathcal{G}$ automatically satisfies the regularity requirement.

The equivalences (i.c)–(i.d) and (ii.c)–(ii.d) follow directly from known results in graph theory. In particular, a connected graph has one single cycle if and only if it has the same number of vertices and edges, and a connected graph has no cycle if and only if it has more vertices than edges (see, for instance, Berge (1962, p. 27ff)).

As an illustration of Theorem 2, notice that the bipartite graph of Figure 4 has two connected cycles, $\langle m_1, w_1, m_3, w_2, m_1 \rangle$ and $\langle m_1, w_2, m_2, w_3, m_1 \rangle$, so the matching is not NTU rationalizable. This also follows immediately from observing that the single connected component of $\mathcal{G}$ involves six vertices and seven edges, so condition (i.d) is not satisfied.

Conditions (i.d) and (ii.d) provide us with a very efficient test to see if a matching is rationalizable: (i) construct the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, (ii) partition this graph into its connected components (e.g., using a depth first search), and (iii) compare the number of vertices and edges on each component. Step (i) takes at most $O(|M| \times |W|)$ steps. Step (ii) is of order $O(|\mathcal{V}| + |\mathcal{E}|)$ and step (iii) also takes linear time. Given that $|\mathcal{E}| \leq |M| \times |W|$, we see that the total running time is $O(|M| \times |W|)$.

4. DISTANCE TO RATIONALIZABILITY

The rationalizability conditions in Theorem 2 only provide a yes/no answer: a matching is either rationalizable or not. In many cases, such a yes/no answer is not very informative as even a small deviation from rationalizability may lead to a rejection of the revealed preference test. As noted by Varian (1990),

[w]hat we usually care about [for revealed preference] is whether optimization is a reasonable way to describe some behavior. For most purposes, ‘nearly optimizing behavior’ is just as good as ‘optimizing’ behavior.

Similar reasoning can be made in the current matching framework: what we really care about is whether the notion of stability provides a reasonably good way to describe the observed matching patterns. As such, we would like to have a measure of how far a given matching is from being rationalizable.⁵

Assume that we have a metric d that captures the distance between two matchings. In particular, for two matchings $\mathcal{M} = (M, W, X)$ and $\mathcal{M}' = (M, W, X')$ with the same set of men and women types, we denote by $d(\mathcal{M}, \mathcal{M}') \geq 0$ the *distance* between $\mathcal{M}$ and $\mathcal{M}'$. A straightforward goodness-of-fit measure is obtained by computing the minimal

⁵In revealed preference analysis, near consistency with rationalizability is usually evaluated using a goodness-of-fit measure. Some examples are Afriat’s critical cost efficiency index Afriat (1973), the Houtman–Maks index (Houtman and Maks (1985)), the Varian index (Varian (1990)), the money pump index (Echenique, Lee, and Shum (2011)), the swaps index (Apesteguia and Ballester (2015)), and the minimum cost index (Dean and Martin (2016)). We are not aware of any other existing goodness-of-fit measure for the current matching context.

distance to the set of rationalizable matchings:

$$\begin{aligned} p(\mathcal{M}) &= \min_{X'} d(\mathcal{M}, \mathcal{M}'), \\ \text{subject to } \mathcal{M}' &= (M, W, X') \text{ is NTU (resp. TU) rationalizable.} \end{aligned} \quad (1)$$

In this paper, we will focus on the L_1 distance:

$$d(\mathcal{M}, \mathcal{M}') = \sum_{m \in M, w \in W} |X(m, w) - X'(m, w)|. \quad (2)$$

In Appendix C, we provide a simple characterization of this distance. We have the following result.

PROPOSITION 3. *Let $\mathcal{M} = (M, W, X)$ be an aggregate matching and let*

$$\begin{aligned} p(\mathcal{M}) &= \min_{X'} \sum_{m \in M, w \in W} (X(m, w) - X'(m, w)) \\ \text{subject to } X'(m, w) &\leq X(m, w), \quad \forall m \in M, w \in W, \quad \text{and} \\ (M, W, X') &\text{ is NTU (resp. TU) rationalizable.} \end{aligned}$$

Then $p(\mathcal{M})$ also solves (1) for the L_1 distance metric defined in (2).

Proposition 3 is proved by showing that the closest rationalizable matching in terms of the L_1 distance will always satisfy $X'(m, w) \leq X(m, w)$ for all m and w . The simple idea behind the proposition is that if a matching is non-rationalizable, then adding matches will never bring us closer to rationalizability.

For ease of interpretation, we will express the result of (1) in terms of the maximal fraction of matchings that one can preserve from $\mathcal{M} = (M, W, X)$ while still guaranteeing that the resulting matching is rationalizable. We call this the *critical matching index* (CMI):

$$\text{CMI}(\mathcal{M}) = 1 - \frac{p(\mathcal{M})}{\sum_{(m, w) \in \mathcal{E}} X(m, w)}.$$

From Proposition 3, we see that the CMI takes values between 0 and 1, and equals 1 if and only if the observed matching is rationalizable. As such, it provides a goodness-of-fit measure of how close a matching is to being rationalizable.

Computing the CMI Using known results from matroid theory, we show that the CMI can be easily computed using a greedy algorithm.⁶ Toward this end, we will focus on

⁶In a previous working paper version (Demuynck and Salman (2020)), we also presented two other procedures to solve this problem. The first takes the form of a linear program that is based on providing an orientation on the edges of the bipartite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The second approach provides an algorithm that is based on the dual of this linear program.

determining the CMI for NTU rationalizability. At the end of this section, we will discuss the modifications that need to take place to compute the CMI for TU rationalizability.

We know from Theorem 2 that a matching $\mathcal{M} = (M, W, X)$ is NTU rationalizable if and only if $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has at most one cycle for each connected component. Such a graph is called a pseudoforest.

For an edge $e = (m, w) \in \mathcal{E}$, let us define the weight $w(e) = X(m, w)$. We are looking for a set of edges $E \subseteq \mathcal{E}$ that has maximum weight and at the same time forms a pseudoforest (forest). This gives the problem

$$q(\mathcal{M}) = \max_{E \subseteq \mathcal{E}} \sum_{e \in E} w(e) \quad \text{subject to} \quad G = (\mathcal{V}, E) \text{ is a pseudoforest.} \quad (3)$$

From Proposition 3, it follows that $q(\mathcal{M}) = \sum_{e \in E} w(e) - p(\mathcal{M})$, so

$$\text{CMI}(\mathcal{M}) = \frac{q(\mathcal{M})}{\sum_{e \in \mathcal{E}} w(e)},$$

To solve (3), we will use matroid theory.⁷ A *matroid* $(Z, \mathcal{I})$ is an algebraic structure that consists of a finite ground set Z of elements and a collection $\mathcal{I}$ of subsets of Z . Elements of $\mathcal{I}$ are called independent sets. A matroid satisfies the following three conditions:

- (i) $\emptyset \in \mathcal{I}$.
- (ii) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
- (iii) If $A, B \in \mathcal{I}$ and $|A| < |B|$, then there is at least one element $a \in B \setminus A$ such that $A \cup \{a\} \in \mathcal{I}$.

The first condition states that the empty set is independent. Condition (ii) requires that any subset of an independent set is independent. The third and most interesting condition states that for any two independent sets of unequal size, it is possible to find an element in the larger set that is not in the smaller one, such that adding this element to the smaller set again gives an independent set. Matroid theory provides a unifying algebraic structure to a wide variety of mathematical concepts. In particular, for a given graph $G = (V, E)$, the pair $(E, \mathcal{I})$, where $\mathcal{I}$ contains all sets $A \subseteq E$ such that (V, A) is a pseudoforest, is also a matroid,⁸ also called the bicircular matroid (see, for example, Simões Pereira (1972) and Matthews (1977)).

⁷See, for example, Oxley (1992) for an introduction to matroid theory.

⁸The first and second conditions are easily checked. To see the third condition, let E and E' be two elements of $\mathcal{I}$ (i.e., (V, E) and (V, E') are pseudoforests) such that $|E| < |E'|$. Let V_E and $V_{E'}$ be the sets of vertices that are part of the edges in E and E' , respectively. As $(V_{E'}, E')$ is a pseudoforest, we have $|V_{E'}| \geq |E'|$. There are two possible cases to consider. If $|V_E| \geq |V_{E'}|$, we obtain $|V_E| \geq |V_{E'}| \geq |E'| > |E|$, so adding one edge from $E' \setminus E$ to E again produces a pseudoforest. On the other hand, if $|V_{E'}| > |V_E|$, then there exists at least one vertex, v , in $V_{E'} \setminus V_E$. Hence, adding an edge from E' having v as an endpoint to E gives a new graph with one more edge and at least one more vertex. This again gives a pseudoforest.

Matroids are closely linked to greedy algorithms.⁹ For a matroid $(Z, \mathcal{I})$ and a non-negative weight function $w : Z \rightarrow \mathbb{R}_+$, we can consider the problem of finding an independent set of maximum weight:

$$\max_{A \subseteq Z} \sum_{a \in A} w(a) \quad \text{subject to} \quad A \in \mathcal{I}. \quad (4)$$

This can be solved using a greedy algorithm. The first step is to sort the elements in Z according to the weights $w(a)$:

$$w(a_1) \geq w(a_2) \geq \dots \geq w(a_n).$$

The algorithm constructs a solution A of (4) by sequentially trying to include the elements $a_1, a_2, \dots, a_n$ while remaining independent. In particular, we start with the empty set $A = \emptyset \in \mathcal{I}$. Next, it is verified whether $\{a_1\} = A \cup \{a_1\} \in \mathcal{I}$. If so, the element a_1 is added to A . If not, the element a_1 is skipped. Next, it is verified whether $A \cup \{a_2\}$ is in $\mathcal{I}$ and if so, a_2 is added to A . In general, at iteration k , we check whether $A \cup \{a_k\}$ is in $\mathcal{I}$. If so, a_k is added to A . If not, a_k is skipped. After n iterations, the resulting set A will solve (4).

Algorithm 1 applies the greedy algorithm to (3). First the edges $e = (m, w) \in \mathcal{E}$ are sorted in descending order according to their weights $w(e) = X(m, w)$. Next, starting from the empty set of edges $E = \emptyset$ and a value $q = 0$, we iterate over the sorted list of edges e and check at each iteration whether $E \cup \{e\}$ is a pseudoforest.

In order to implement Algorithm 1, we need to find an efficient way to verify whether at each iteration, $E \cup \{e\}$ is a pseudoforest. For this, we can use the disjoint-set data structure (Galler and Fisher (1964)). This data structure stores a partition of a finite set of elements and can determine membership (i.e., find the set to which a particular element belongs), and can merge two sets (union) in log time.

Algorithm 1 The Greedy Algorithm.

- 1: Sort the edges $\mathcal{E}$ in descending order according to the weight function $w : \mathcal{E} \rightarrow \mathbb{R}_+$, where for $e = (m, w)$

$$w(e) = X(m, w).$$

- 2: Initialize $E = \emptyset$ and $q = 0$.
 - 3: **for** $e = (m, w)$ over the sorted list of edges **do**
 - 4: **if** $E \cup \{e\}$ is a pseudoforest **then**
 - 5: $E \leftarrow E \cup \{e\}$
 - 6: $q \leftarrow q + w(e)$
 - 7: **end if**
 - 8: **end for**
-

⁹See, for example, Oxley (1992, Section 1.8).

Our disjoint-set data structure will contain the connected components (in terms of its vertices) of the graph $(\mathcal{V}, E)$, together with an identifier that tells us whether the component is a tree or not. At a certain iteration of the algorithm, if edge $e = (m, w)$ is selected from the sorted list, we first use two membership calls to find the components of $(\mathcal{V}, E)$ that contain m and w . If the components are the same and if it is a tree, $w(e)$ is added to q and the component is marked as being no longer a tree (as adding the edge e creates a cycle in the component).

If m and w belong to distinct components and if both components are a tree, then $w(e)$ is added to q , the components are merged together, and the resulting component is labelled as a tree. If m and w are in distinct components and if only one of those is a tree, then $w(e)$ is added to q , and the two components are merged, and marked as not being a tree (as the merged component will have a cycle). In any other case, the edge e is skipped. Sorting the edges takes $O(|\mathcal{E}| \log(|\mathcal{E}|))$ steps (e.g., using merge sort). For each edge, we need to perform two membership calls and possibly one merge. Each of these steps take $O(\log(|\mathcal{V}|))$ time, so the entire algorithm has running time $O(|\mathcal{E}| \log(|\mathcal{E}|))$.¹⁰

The output of Algorithm 1 gives a matching $\mathcal{M}' = (M, W, X')$ that solves (3).

TU rationalizability In order to compute the CMI for TU rationalizability, we only need to make a few modifications. From Theorem 2, we know that a matching is TU rationalizable if the graph $\mathcal{G}$ has no cycles. A graph without any cycles is known as a forest.

The collection of subforests of a given graph is also a matroid, called the cycle matroid. As such, we can easily modify Algorithm 1 to the TU setting simply by replacing the concept of pseudoforests by its forest analogue. The single change for the algorithm is in step 4, which we have to modify to

if $E \cup \{e\}$ is a forest then.

This modified algorithm is identical to finding a maximum spanning tree of $\mathcal{G}$, which is a well known problem in computer science (Kruskal (1956)).

5. REMOVING TYPES

In this section, we look at another way to measure the distance from a given matching to the set of rationalizable matchings.

Let us call $\mathcal{M}' = (M, W', X')$ a *woman-type reduction* of the matching $\mathcal{M} = (M, W, X)$ if $W' \subseteq W$, and for all $m \in M$ and $w \in W'$, $X'(m, w) = X(m, w)$. In other words, it is the matching obtained by removing all woman types in $W \setminus W'$.¹¹

Given a matching $\mathcal{M} = (M, W, X)$ that is not NTU or TU rationalizable, we would like to find the largest woman-type reduction $\mathcal{M}'$ of $\mathcal{M}$, in terms of $|W'|$, such that $\mathcal{M}'$ is

¹⁰See also Gabow and Tarjan (1988) for a similar but even more efficient linear time algorithm. However, their procedure relies on some additional, more intricate, data structures.

¹¹In a similar way, we can also define a man-type reduction (by restricting M' to be a subset of M) or a joint type reduction (when both M' and W' are subsets of M and W , respectively). For simplicity however, we will focus on woman-type reductions. However, the proof of Proposition 4 below can easily be adjusted to these other settings.

rationalizable:

$$\max_{W' \subseteq W} |W'| \quad \text{subject to} \quad \mathcal{M}' = (M, W', X') \text{ is NTU (resp. TU) rationalizable.} \quad (5)$$

It is easy to see that if $|W'| = 1$, then the woman-type reduction $\mathcal{M}'$ is NTU and TU rationalizable as the corresponding graph $\mathcal{G}$ will have no cycles. As such, (5) is well defined.

Let us first reformulate this problem as a decision problem.

DECISION PROBLEM (MTR-NTU: Maximum Type Reduction for NTU). Given a matching $\mathcal{M} = (M, W, X)$ and a number $K \in \mathbb{N}$, does there exist a woman-type reduction $\mathcal{M}' = (M, W', X')$ such that $\mathcal{M}'$ is NTU rationalizable and $|W'| \geq K$?

DECISION PROBLEM (MTR-TU: Maximum Type Reduction for TU). Given a matching $\mathcal{M} = (M, W, X)$ and a number $K \in \mathbb{N}$, does there exist a woman-type reduction $\mathcal{M}' = (M, W', X')$ such that $\mathcal{M}'$ is TU rationalizable and $|W'| \geq K$?

We have the following result.

PROPOSITION 4. *The decision problems MTR-NTU and MTR-TU are both NP-complete.*

This result shows that it is not possible (unless $P = NP$) to find an efficient (polynomial time) algorithm to solve (5).¹² The proof of the proposition uses a reduction from the maximum independent set problem.¹³

DECISION PROBLEM (MIS: Maximum Independent Set). Given a graph $G = (V, E)$ and a number $k \in \mathbb{N}$, is there a set of vertices $\tilde{V} \subseteq V$ such that $|\tilde{V}| \geq k$ and no two vertices in $\tilde{V}$ are connected by an edge in E .

6. A MARRIAGE MARKET ILLUSTRATION

For our illustration, we look at a marriage market setting and focus on the NTU rationalizability condition.¹⁴ We use the data set from Dupuy and Galichon (2014). This is data from the 1993–2002 waves of the DNB household survey undertaken by CentERdata.¹⁵ The data are a representative panel of the Dutch population.¹⁶ The sample contains information on several characteristics of both spouses for 1158 couples. In order to analyze the rationalizability of this matching, we first need to define the types of men and women. We do this on the basis of a subset of observable characteristics (see Table 2 in Appendix B for summary statistics).

¹²See Garey and Johnson (1979) for an excellent introduction to the theory of computational complexity and NP hardness in particular.

¹³This is problem GT20 in Garey and Johnson (1979).

¹⁴In the working paper version, we conducted a more thorough empirical exercises. For interested readers, see, Demuynck and Salman (2020).

¹⁵See <https://www.eui.eu/Research/Library/ResearchGuides/Economics/Statistics/DataPortal/DNB>.

¹⁶We refer to the paper of Dupuy and Galichon (2014) for a description of the data set and the details on the procedure used to generate the final sample. The full data set and documentation can be downloaded from <https://doi.org/10.1086/677191>.

- **Education.** We follow Dupuy and Galichon (2014) and consider the following three categories: (i) lower education (kindergarten, primary, elementary, secondary); (ii) intermediate (secondary, pre-university, vocational); (iii) higher education (university).
- **BMI** (Body Mass Index). This variable is discretized using the cutoffs from the World Health Organization:¹⁷ (a) if the BMI is below 18.5 (underweight), (b) if the BMI is between 18.5 and 24.99 (normal), (c) if the BMI is between 25 and 30 (overweight), and (d) if the BMI is above 30 (obese).
- **Height.** This variable is categorized using the mean μ and standard deviation σ of the height of all men (resp. women). We categorize the height variable equal to (a) if height is below $\mu - \sigma$, (b) if the height is between $\mu - \sigma$ and μ , (c) if the height is between μ and $\mu + \sigma$, and (d) if the height is above $\mu + \sigma$.
- **Age.** We discretize this variable using 3 year intervals.

For the illustration, we define types by considering combinations of three or four of the characteristics given above.

Table 1 gives the results. The first row gives the CMI for the matching. This measures how far the data set is from being rationalizable. For example, the number 0.394 shows that if we define types on the basis of education, height, and BMI, we need to remove almost 60% of all 1158 matchings in order to obtain a matching that is rationalizable.

The second row in the table gives the p -value for the null hypothesis that the observed matching is random. This is conducted by sampling a large number of random matchings with, for each type, the same number of men and women as in the observed matching.¹⁸ The p -value is the fraction of these random matchings that have a CMI that is larger than or equal to the CMI of the observed matching. We refer to the working paper version for more detailed information (Demuynck and Salman (2020)). The last two lines provide information on the number of man types and woman types in the matching.

TABLE 1. Figures for marriage market illustration.

Type Dimensions	Education Height BMI	Education Height Age	Education BMI Age	Height BMI Age	Education Height BMI
CMI	0.394	0.391	0.472	0.384	0.472
p -value	0.0004	0.0	0.0	0.0	0.0
$ M $	47	175	167	232	697
$ W $	48	83	83	109	330

¹⁷See <https://www.euro.who.int/en/health-topics/disease-prevention/nutrition/a-healthy-lifestyle/body-mass-index-bmi> (accessed on July 7, 2020).

¹⁸For this, we follow the algorithm of Agresti, Wackerly, and Boyett (1979). The p -values are computed on the basis of 17,000 random matches.

7. CONCLUSION

This paper complements and extends the revealed preference analysis in [Echenique et al. \(2013\)](#). We provide an efficient way to test whether aggregate matchings are TU or NTU rationalizable by comparing the number of vertices and edges of the connected components of a bipartite graph. We further study non-rationalizable matchings. We draw the connection with matroid theory to obtain an efficient algorithm that permits the identification of the minimum number of matches that need to be removed to restore rationalizability and we show that restoring rationalizability by removing a minimal number of types is NP-hard. We provided an short illustration to show the practical relevance of using the revealed preference approach to analyze aggregate matchings.

From an empirical point, the results of this paper bring us closer to the verification of stability for matching markets where preferences of the individuals are unobservable, which is the case in most real life decentralized matching markets. Our illustration show that using goodness-of-fit measures like the CMI makes it possible to capture the degree of non-rationalizability in these markets.

The fact that the set of TU or NTU rationalizable submatchings forms a matroid might be particularly interesting for future research. Matroid theory is a vast and expanding area of research in mathematics and computer science. In this regard, several other results from this field may also be relevant for the current framework. As an example, [Edmonds \(1965\)](#) constructed an efficient algorithm that can partition a given set into a minimum number of independent subsets. Translating this to our framework, it means that we can efficiently decompose a given matching into a minimal number of rationalizable matchings.

APPENDIX A: PROOFS OF RESULTS IN MAIN TEXT

A.1 Proof of Theorem 2

The equivalence (i.a)–(i.b) and (ii.a)–(ii.b) follow from Theorem 1. Also, the equivalences (i.c)–(i.d) and (ii.c)–(ii.d) follow from known results in the literature (see [Berge \(1962, p. 27ff\)](#)). As such, we will focus on proving the equivalence (i.b)–(i.c) and (ii.b)–(ii.c). For this, we first state some intermediate results.

LEMMA 5. *Let $\gamma = \langle v_0, \dots, v_n, v_0 \rangle$ be a cycle. Then there exists no strict subset of edges of γ that also form a cycle.*

PROOF. Toward a contradiction, assume that γ' is a cycle formed from a subset of edges of γ . Let (V, E) be the graph consisting of the vertices and edges of γ , and let (V', E') be the graph formed by the set of vertices and edges in γ' .

As every vertex and edge in a cycle is distinct, it follows that every vertex in V is part of exactly two edges in E , and every vertex in V' is part of exactly two edges in E' .

As $V' \subset V$ and (V, E) is connected, there must be an edge $e \in E$ that connects some vertex $v' \in V'$ to a vertex in $V \setminus V'$. This, however, implies that $e \notin E'$, so v' has only one edge in (V', E') , a contradiction. $\square$

Fix a natural number n . For a natural numbers $i \leq n$, I write $\lfloor i \rfloor$ to denote $i \bmod (n+1)$ (i.e., the remainder of dividing i by $n+1$).

LEMMA 6. *If $i = \lfloor j+1 \rfloor$ and $j = \lfloor i+1 \rfloor$, then $n = 0$ or $n = 1$.*

PROOF. The first condition requires that there is a $q \in \mathbb{N}$ such that $j+1 = q(n+1) + i$. Similarly, the second condition requires that there is an $s \in \mathbb{N}$ such that $i+1 = s(n+1) + j$. Solving this system gives

$$2 = (q+s)(n+1).$$

As $q, s, n \in \mathbb{N}$, this implies that either $n = 0$ or $n = 1$. $\square$

As a first result, we demonstrate the intuitive fact that every regular, vertex-minimal cycle in the graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ should have no three consecutive vertices on a common horizontal or vertical line.

LEMMA 7. *Consider a matching $\mathcal{M} = (M, W, N)$ with graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$. If $\gamma = \langle v_0, \dots, v_n, v_0 \rangle$ is a regular, vertex-minimal cycle in $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, then for any three subsequent vertices $v_i = (m_i, w_i)$, $v_{\lfloor i+1 \rfloor} = (m_{\lfloor i+1 \rfloor}, w_{\lfloor i+1 \rfloor})$, and $v_{\lfloor i+2 \rfloor} = (m_{\lfloor i+2 \rfloor}, w_{\lfloor i+2 \rfloor})$, the following statements hold:*

- *if $m_i = m_{\lfloor i+1 \rfloor}$, then $w_{\lfloor i+1 \rfloor} = w_{\lfloor i+2 \rfloor}$,*
- *if $w_i = w_{\lfloor i+1 \rfloor}$, then $m_{\lfloor i+1 \rfloor} = m_{\lfloor i+2 \rfloor}$.*

PROOF. Let $\gamma = \langle v_0, v_1, \dots, v_n, v_0 \rangle$ be a regular, vertex-minimal cycle and consider three consecutive vertices $v_i = (m_i, w_i)$, $v_{\lfloor i+1 \rfloor} = (m_{\lfloor i+1 \rfloor}, w_{\lfloor i+1 \rfloor})$, and $v_{\lfloor i+2 \rfloor} = (m_{\lfloor i+2 \rfloor}, w_{\lfloor i+2 \rfloor})$ in the cycle. Given that the cycle is regular, it should have at least four distinct vertices. As such, $\gamma \neq \langle v_i, v_{\lfloor i+1 \rfloor}, v_{\lfloor i+2 \rfloor}, v_i \rangle$.

Now, toward a contradiction, assume that $m_i = m_{\lfloor i+1 \rfloor}$ and $w_{\lfloor i+1 \rfloor} \neq w_{\lfloor i+2 \rfloor}$. Given that $(v_{\lfloor i+1 \rfloor}, v_{\lfloor i+2 \rfloor}) \in \mathbb{E}$, it must be that $m_{\lfloor i+1 \rfloor} = m_{\lfloor i+2 \rfloor}$. As such, $m_i = m_{\lfloor i+2 \rfloor}$ and, therefore, $(v_i, v_{\lfloor i+2 \rfloor}) \in \mathbb{E}$, a contradiction with minimality of γ .

The case where $w_i = w_{\lfloor i+1 \rfloor}$ and $m_{\lfloor i+1 \rfloor} \neq m_{\lfloor i+2 \rfloor}$ leads to a similar contradiction. $\square$

If $\gamma = \langle v_0, \dots, v_n, v_0 \rangle$ is a cycle in $\mathbb{G}$ and if for some type $m \in M$, there is an i such that $v_i = (m, w)$, we say that the cycle γ visits m . We use a similar definition for visiting a women type w . The following lemma shows that every regular, vertex-minimal cycle of $\mathbb{G}$ visits a type at most twice and this on subsequent vertices.

LEMMA 8. *If $\gamma = \langle v_0, \dots, v_n, v_0 \rangle$ is a regular, vertex-minimal cycle in $\mathbb{G}$, then for all $m \in M$ ($w \in W$) that are visited by γ , there are exactly two subsequent vertices v_i and $v_{\lfloor i+1 \rfloor}$ in the cycle that contain m (w) and no other vertex contains m (w).*

PROOF. We show the proof for a man type m . The proof for a women type w is entirely similar.

Consider a regular, vertex-minimal cycle $\gamma = \langle v_0, v_1, \dots, v_n, v_0 \rangle$ that visits m . This means that there is a vertex $v_i = (m, w)$ for some $w \in W$. As $(v_{\lfloor i-1 \rfloor}, v_i) \in \mathbb{E}$, it must be that either $m_{\lfloor i-1 \rfloor} = m$ or $w_{\lfloor i-1 \rfloor} = w$. If the latter is the case, then from Lemma 7, it follows that $m = m_{\lfloor i+1 \rfloor}$. As such, either $v_{\lfloor i-1 \rfloor}$ contains m or $v_{\lfloor i+1 \rfloor}$ contains m . This shows that there are at least two subsequent vertices in γ that contain m . Toward a contradiction, assume that there are three vertices that contain m , say

$$v_i = (m, w_i), \quad v_j = (m, w_j), \quad v_k = (m, w_k)$$

for some $w_i, w_j, w_k \in W$. Without loss of generality, assume that $0 \leq i < j < k \leq n$. If i, j and k are three consecutive numbers, then by Lemma 7, we have $w_j = w_k$, which implies that $v_j = v_k$. Given that the only identical vertices in a cycle are the first and last vertices, this gives $j = k = 0$, a contradiction.

Given this, assume, without loss of generality, that $j + 1 < k$. However, then $\langle v_0, \dots, v_j, v_k, \dots, v_n, v_0 \rangle$ is a cycle with fewer vertices, contradicting the vertex-minimality of γ . $\square$

The following lemma shows that there is a one to one correspondence between regular, vertex-minimal cycles in $\mathbb{G}$ and cycles in $\mathcal{G}$. As a corollary, we immediately obtain the equivalence between conditions (ii.b) and (ii.c) of Theorem 2.

LEMMA 9. *For $n \geq 3$, the following statements hold:*

- (i) *If $\gamma = \langle v_0, v_1, \dots, v_n, v_0 \rangle$ is a regular, vertex-minimal cycle in $\mathbb{G}$ with $v_i = (m_i, w_i)$ and $w_0 = w_1$, then $\gamma' = \langle m_0, w_1, m_2, \dots, m_{n-1}, w_n, m_0 \rangle$ is a cycle in $\mathcal{G}$.*
- (ii) *If $\gamma = \langle v_0, v_1, \dots, v_n, v_0 \rangle$ is a regular, vertex-minimal cycle in $\mathbb{G}$ with $v_i = (m_i, w_i)$ and $m_0 = m_1$, then $\gamma' = \langle w_0, m_1, w_2, \dots, w_{n-1}, m_n, w_0 \rangle$ is a cycle in $\mathcal{G}$.*
- (iii) *If $\gamma' = \langle m_0, w_1, m_2, \dots, m_{n-1}, w_n, m_0 \rangle$ is a cycle in $\mathcal{G}$, then $\gamma' = \langle v_0, v_1, \dots, v_n, v_0 \rangle$ with $v_i = (m_i, w_{i+1})$ if i is even and $v_i = (w_i, m_{\lfloor i+1 \rfloor})$ if i is odd is a regular, vertex-minimal cycle in $\mathbb{G}$.*
- (iv) *If $\gamma' = \langle w_0, m_1, w_2, \dots, w_{n-1}, m_n, w_0 \rangle$ is a cycle in $\mathcal{G}$, then $\gamma' = \langle v_0, v_1, \dots, v_n, v_0 \rangle$ with $v_i = (w_i, m_{i+1})$ if i is even and $v_i = (m_i, w_{\lfloor i+1 \rfloor})$ if i is odd is a regular, vertex-minimal cycle in $\mathbb{G}$.*

PROOF. We are going to demonstrate (i) and (iii). The two other statements are analogues.

Recall that a sequence of vertices $\langle x_0, \dots, x_n, x_0 \rangle$ is a cycle in the graph $G = (V, E)$ if

- for all i , $(x_i, x_{\lfloor i+1 \rfloor}) \in E$
- all vertices $x_0, \dots, x_n$ are distinct
- all edges $(x_i, x_{\lfloor i+1 \rfloor})$ are distinct.

For (i), let $\gamma = \langle v_0, v_1, \dots, v_n, v_0 \rangle$ be a regular, vertex-minimal cycle in $\mathbb{G}$ with $v_i = (m_i, w_i)$ and $w_0 = w_1$.

Then, from repeated application of Lemma 7, we see that $w_0 = w_1$, $m_1 = m_2$, and $w_2 = w_3, \dots, m_n = m_0$. This shows that $v_i = (w_i, m_i) = (w_i, m_{\lfloor i+1 \rfloor}) \in \mathcal{E}$ for i odd and $v_i = (m_i, w_i) = (m_i, w_{i+1}) \in \mathcal{E}$ for i even. Also notice that n is odd. Let us show that

$$\gamma' = \langle m_0, w_1, m_2, \dots, w_n, m_0 \rangle$$

is a cycle in $\mathcal{G}$. For this, we need to show that γ' has no identical edges and $m_0, w_1, \dots, m_{n-1}, w_n$ does not contain identical vertices.

To show that γ' has no identical vertices, assume, toward a contradiction, that $m_i = m_j$ for $i \neq j$ in γ' . Notice that both i and j are even. Then from the definition, $v_i = (m_i, w_{i+1})$ and $v_j = (m_j, w_{j+1})$. As $m_i = m_j$, Lemma 9 tells us that v_i and v_j are adjacent, i.e., $i = \lfloor j+1 \rfloor$ or $j = \lfloor i+1 \rfloor$. Lemma 6 shows that $n = 0$ or $n = 1$. As n is odd, we have $n = 1$, so $\gamma = \langle v_0, v_1, v_0 \rangle$, which contradicts regularity.

Next, let us show that γ' has no two identical edges. If $(m_i, w_{i+1}) = (m_j, w_{j+1})$ with $i \neq j$, this contradicts the fact that $m_0, \dots, w_n$ cannot have two identical vertices. The same holds if $(w_i, m_{\lfloor i+1 \rfloor}) = (w_j, m_{\lfloor j+1 \rfloor})$ for some $i \neq j$. Next, if $(m_i, w_{i+1}) = (m_{\lfloor j+1 \rfloor}, w_j)$, then by a similar reasoning it must be that $i = \lfloor j+1 \rfloor$ and $i+1 = j$. Lemma 6 tells us that $n = 1$ or $n = 0$. However, n is odd, so $n = 1$. This gives $\gamma = \langle v_0, v_1, v_0 \rangle$, which again contradicts regularity.

For (iii), consider a cycle $\gamma' = \langle m_0, w_1, m_2, \dots, m_{n-1}, w_n, m_0 \rangle$ in $\mathcal{G}$. Notice that n is an odd number. Consider the vertices $v_i = (m_i, w_{i+1})$ for i even and $v_i = (w_i, m_{\lfloor i+1 \rfloor})$ for i odd in $\mathbb{G}$. Then for all i , $(v_i, v_{\lfloor i+1 \rfloor}) \in \mathbb{E}$. Let us show that

$$\gamma = \langle v_0, v_1, \dots, v_n, v_0 \rangle$$

is a regular, vertex-minimal cycle in $\mathbb{G}$. First notice that $\mathbb{G}$ visits at least two man and two woman types, so it is regular. Next, as γ' is a cycle, every man type m and woman type w is contained in at most two vertices of γ (if not, then γ' would contain at least three edges involving the same man or woman type, which means that γ' has two identical vertices). In order to show that γ is a cycle, we need to show that γ has no identical vertices and no identical edges.

Toward a contradiction, assume that $v_i = v_j$ with $i \neq j$. If both i and j are even, then $(m_i, w_{i+1}) = (m_j, w_{j+1})$, which implies that γ' has two identical edges, a contradiction. If both i and j are odd, a similar contradiction occurs. If i is even and j is odd, then

$$(m_i, w_{i+1}) = v_i = v_j = (m_{\lfloor j+1 \rfloor}, w_j).$$

Given that all vertices in $m_0, w_1, \dots, w_n$ are distinct, this gives $i = \lfloor j+1 \rfloor$ and $i+1 = j$. From Lemma 6, we know that $n = 1$ or $n = 0$. As n is odd, we have $n = 1$ and, therefore, $\gamma' = \langle m_0, w_1, m_0 \rangle$, which contradicts the fact that a cycle cannot have two identical edges. The case where i is odd and j is even is similar.

In order to see that $\mathbb{G}$ does not contain identical edges, assume toward a contradiction that $(v_i, v_{\lfloor i+1 \rfloor}) = (v_j, v_{\lfloor j+1 \rfloor})$ with $i \neq j$. Then either $(v_i = v_j$ and $v_{\lfloor i+1 \rfloor} = v_{\lfloor j+1 \rfloor})$

or $(v_i = v_{\lfloor j+1 \rfloor} \text{ and } v_{\lfloor i+1 \rfloor} = v_j)$. The first case contradicts the first part of the proof as it would mean that $v_0, \dots, v_n$ has two identical vertices. As such it must be that $v_{\lfloor i+1 \rfloor} = v_j$ and $v_{\lfloor j+1 \rfloor} = v_i$. Given that all vertices are distinct, it follows that $i = \lfloor j+1 \rfloor$ and $\lfloor i+1 \rfloor = j$. Again, by Lemma 6, $n = 1$ or $n = 2$ and the latter violates the condition that n is odd. As such, $\gamma' = \langle m_0, w_1, m_0 \rangle$, which contradicts the fact that γ' has no two identical edges.

Finally, we need to show that γ is a vertex-minimal cycle. If not, there exists a subset of vertices of γ that also forms a cycle. Let $\tilde{\gamma}$ be such a cycle. Without loss of generality, we can assume that $\tilde{\gamma}$ is vertex-minimal. If this cycle is regular, then from part (i) of the proof, this produces a cycle in $\mathcal{G}$, say $\tilde{\gamma}'$, whose set of edges is a strict subset of the edges from γ' . This, however, contradicts Lemma 5. If the cycle is not regular, i.e., only contains one man type or one woman type, then as $\tilde{\gamma}$ contains at least three vertices, there should be a man or woman type that was involved in more than two vertices of γ , a contradiction. $\square$

The following lemma shows the equivalence between (i.b) and (i.c).

LEMMA 10. *The graph $\mathbb{G}$ has two distinct regular, vertex-minimal connected cycles if and only if the bipartite graph $\mathcal{G}$ has two distinct connected cycles.*

PROOF. Let γ_1 and γ_2 be two regular, vertex-minimal connected cycles in $\mathbb{G}$, and let γ'_1 and γ'_2 be the corresponding cycles in $\mathcal{G}$ (see Lemma 9). Given the bijection between such vertex-minimal cycles in $\mathbb{G}$ and cycles in $\mathcal{G}$, we have that γ_1 and γ_2 are distinct if and only if γ'_1 and γ'_2 are distinct.

($\Rightarrow$) Assume that γ_1 and γ_2 are distinct and connected in $\mathbb{G}$. If γ_1 and γ_2 have a vertex in common, then γ'_1 and γ'_2 have a common edge, so we are done.

If not, let $\rho = \langle v_0, \dots, v_n \rangle$ be a path in $\mathbb{G}$ between the cycles γ_1 and γ_2 . Without loss of generality, assume that ρ is vertex-minimal (i.e., there is no subset of vertices in ρ that also connects v_0 to v_n). Similar to the proof of Lemma 7, we can show that ρ has no three subsequent vertices on a common horizontal or vertical line, i.e., for $v_i = (m_i, w_i)$, $v_{i+1} = (m_{i+1}, w_{i+1})$ and $v_{i+2} = (m_{i+2}, w_{i+2})$, if $m_i = m_{i+1}$, then $w_{i+1} = w_{i+2}$, and if $w_i = w_{i+1}$, then $m_{i+1} = m_{i+2}$.

Assume that $w_0 = w_1$ and n is odd. The other cases can be treated in a similar way. Notice that $m_0 \in \gamma'_1$ and $w_n \in \gamma'_2$. As such,

$$\rho' = \langle m_0, w_1, m_2, w_3, \dots, m_{n-1}, w_n \rangle$$

has $(m_i, w_{i+1}) \in \mathcal{E}$, $m_0 \in \gamma'_1$, and $w_n \in \gamma'_2$. Given that ρ is a vertex-minimal path, we can use a proof, very similar to the proof of Lemma 9, to show that ρ' is also a path. This shows that γ'_1 and γ'_2 are connected.

For the reverse, let γ'_1 and γ'_2 be two cycles in $\mathcal{G}$. Let γ_1 and γ_2 be the corresponding vertex-minimal cycles in $\mathbb{G}$ (see Lemma 9). If the two cycles γ'_1 and γ'_2 have a vertex in common, say m , then there is an edge $v = (m, w)$ in γ'_1 and an edge $v' = (m, w')$ in γ'_2 . Also, v is in γ_1 and v' is in γ_2 . As $(v, v') \in \mathbb{E}$, we have that γ_1 and γ_2 are connected.

Now assume that γ'_1 and γ'_2 are disjoint. Let ρ be a path of length n from γ'_1 to γ'_2 . Without loss of generality, we can assume that ρ has minimal length. Also assume that n is odd. The case where n is even is treated in a similar way. Then

$$\rho' = \langle m_0, w_1, \dots, w_n \rangle.$$

Define $v_i = (m_i, w_{i+1})$ for i even and $v_i = (w_i, m_{i+1})$ for $i < n$ odd. We see that for all i , $(v_i, v_{i+1}) \in \mathbb{E}$ and as ρ' is a path, we have that $v_i \neq v_j$ for all i, j . Then

$$\rho = \langle v_0, \dots, v_{n-1} \rangle$$

is a path in $\mathbb{G}$. If $v_0 = (m_0, w_1) \in \gamma_1$ and $v_{n-1} = (m_{n-1}, w_n) \in \gamma_2$, this gives a path from γ_1 to γ_2 . If $v_0 = (m_0, w_1) \notin \gamma_1$ and $v_{n-1} = (m_{n-1}, w_n) \in \gamma_2$, then as $m_0 \in \gamma'_1$, there is a $w' \in \gamma'_1$ such that (w, m_0) is an edge of γ'_1 . Define $v = (w, m_0)$ to be the corresponding vertex of γ_1 . Then

$$\langle v, v_0, \dots, v_{n-1} \rangle$$

is a path in $\mathbb{G}$ from γ_1 to γ_2 . The other cases can be dealt with in a similar way. $\square$

A.2 Proof of Proposition 3

Rewriting (1) using the distance function (2) gives

$$\begin{aligned} \min_{X'} \sum_{m \in M, w \in W} |X(m, w) - X'(m, w)| \\ \text{subject to } (M, W, X') \text{ is rationalizable.} \end{aligned} \quad (6)$$

The following lemma shows that any solution X' to this problem has fewer matches than X .

LEMMA 11. *Any solution X' to (6) has that $X'(m, w) \leq X(m, w)$ for all $m \in M$ and $w \in W$.*

PROOF. Assume that for some $\hat{m} \in M$ and $\hat{w} \in W$, $X'(m, w) > X(m, w)$. Consider the new matching $(M, W, \hat{X}(m, w))$, where for all m, w with $m \neq \hat{m}$ or $w \neq \hat{w}$, $\hat{X}(\hat{m}, \hat{w}) = X'(m, w)$ and $\hat{X}(\hat{m}, \hat{w}) = X(\hat{m}, \hat{w})$. We have that $(M, W, \hat{X})$ is also rationalizable as it contains, for all types, no more matches compared to X' (so the bipartite graph of $(M, W, \hat{X})$ will be a subgraph of the bipartite graph of (M, W, X')). Also

$$\begin{aligned} \sum_{m \in M, w \in W} |X(m, w) - X'(m, w)| \\ = \sum_{m \in M, w \in W} |X(m, w) - \hat{X}(m, w)| + |X(\hat{m}, \hat{w}) - X'(\hat{m}, \hat{w})| \\ > \sum_{m, w} |X(m, w) - \hat{X}(m, w)|, \end{aligned}$$

which shows that X' was not optimal, a contradiction. $\square$

Lemma 11 shows that (6) can be rewritten as

$$\begin{aligned} \min_{X'} \quad & \sum_{m \in M, w \in W} (X_{m,w} - X'_{m,w}) \\ \text{subject to} \quad & (M, W, X') \text{ is rationalizable and} \\ & X'_{m,w} \leq X_{m,w}, \quad \forall m \in M, \forall w \in W. \end{aligned}$$

which is the statement of Proposition 3 in the main text.

A.3 Proof of Proposition 4

Given a woman-type reduction $\mathcal{M}' = (M, W', X')$ and a number K , it is easy to verify (i) whether $|W'| \geq K$ and (ii) whether the reduction is NTU or TU rationalizable (using Proposition 2) in polynomial time. This shows that the problem MTR-NTU and MTR-TU is in NP time.

In order to show that the problem is NP-hard, we use a reduction from the problem MIS: Given a graph $G = (V, E)$ and a number $k \in \mathbb{N}$, is there a set of vertices $\tilde{V} \subseteq V$ such that $|\tilde{V}| \geq k$ and no two vertices in $\tilde{V}$ are connected by an edge in E .

We need to show that for every instance of MIS we can construct (in polynomial time) an instance of MTR-NTU and MTR-TU such that the instance for the MIS problem is a yes if and only if the instance for the associated MTR problem is a yes. We will first prove this for MTR-NTU and then for MTR-TU.

MTR-NTU Consider the graph $G = (V, E)$ and a number $k \in \mathbb{N}$ as an instance of MIS. We construct an instance of MTR-NTU, i.e., a matching (M, W, X) and a number K in the following way:

- For every vertex $v \in V$, we construct two woman types $w_{v,1}$ and $w_{v,2}$.
- For every $v \in V$, we consider two man-types $m_{v,1}$ and $m_{v,2}$.
- For every vertex $v \in V$ and every edge $e \in E$ connected to v , we construct a man type $m_{v,e}$.

To define the matchings among these types, we set

$$\begin{aligned} X(m_{v,1}, w_{v,1}) &= 1 \\ X(m_{v,1}, w_{v,2}) &= 1 \\ X(m_{v,2}, w_{v,1}) &= 1 \\ X(m_{v,2}, w_{v,2}) &= 1 \\ X(m_{v,e}, w_{v,2}) &= 1 \quad \text{if } e \text{ is adjacent to } v. \end{aligned}$$

Figure 5 gives the example of the configuration for a vertex v with three adjacent edges, e_1 , e_2 , and e_3 . Notice that this constitutes a cycle. As such, we call it the cycle of the vertex v or v cycle.

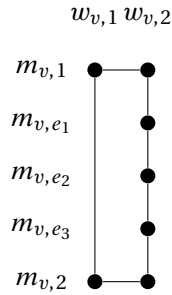


FIGURE 5. Illustration of a cycle of a vertex v with three adjacent edges e_1 , e_2 , and e_3 .

A next part of the MTR instance connects the various v cycles with each other.

- For every edge $e = (v_i, v_j) \in E$, we construct a woman type w_e and we set

$$X(m_{v_i,e}, w_e) = 1$$

$$X(m_{v_j,e}, w_e) = 1.$$

To finish the description of the MTR-NTU instance, we set the parameter K for the MTR-NTU problem equal to $|V| + |E| + k$.

This finishes the instance construction. In total, we have $2|V| + \sum_v d(v)$ man types and $2|V| + |E|$ woman types, where d is the degree of the vertex v . This number is bounded by $4|V| + |V|^2 + |E|$, which is polynomial in the inputs.

As an illustrative example of how the instance of MTR-NTU may look, consider the network given in Figure 6, which has four vertices $\{a, b, c, d\}$ and four edges $\{e_1, e_2, e_3, e_4\}$.

The MTR-NTU instance corresponding to this graph is given in Figure 7.

In order to finish the proof, we need to show that every yes instance of MIS corresponds to a yes instance for the associated MTR-NTU problem and vice versa.

Toward this end, let $\{G = (V, E), k\}$ be the instance of the MIS problem and let $\{\mathcal{M} = (M, W, X), K\}$ be the associated instance of the MTR-NTU problem. Assume that $\{G = (V, E), k\}$ is a yes instance and let $\tilde{V}$ be an independent set of size $|\tilde{V}| \geq k$. Let us show that there is a woman-type reduction (M, W', X') of size $|W'| \geq K = |V| + |E| + k$.

For all $v \in V \setminus \tilde{V}$, remove from W the type $w_{v,2}$. Notice that this destroys the cycle of vertex v . This leaves us with at least $|V| + |E| + k = K$ woman types W' . Now assume that $\mathcal{M}' = (M, W', X')$ is not rationalizable. In this case, there should be two distinct

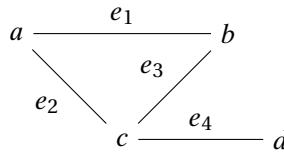


FIGURE 6. An example graph $G = (V, E)$ with four vertices and four edges.

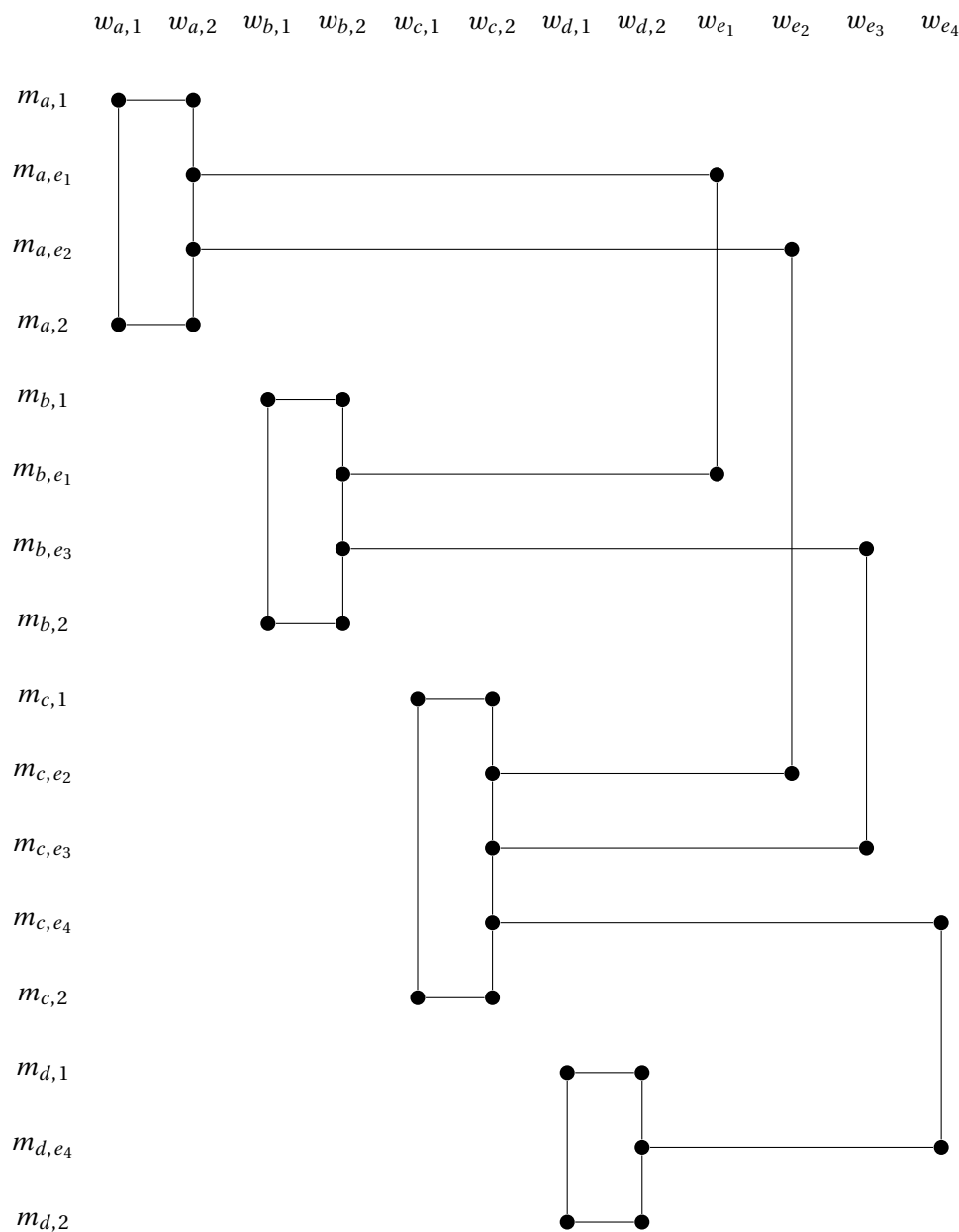


FIGURE 7. MTR-NTU instance of the graph in Figure 6.

connected minimal cycles. However, this can only be if two v cycles with $v \in \tilde{V}$ are still connected. This contradicts the definition of an independent set.

For the reverse, let $\{\mathcal{M} = (M, W, X), K\}$ be a yes instance for MTR-NTU and let $\mathcal{M}' = (M, W', X')$ be a woman-type reduction with $|W'| \geq K = |V| + |E| + k$. We are going to perturb this solution slightly to a new yes instance of MTR-NTU in the following way:

- P1. For all w , if $w \in W \setminus W'$ and $w = w_{v,1}$ for some vertex $v \in V$, and if $w_{v,2} \in W'$, we remove $w_{v,2}$ from W' and add $w_{v,1}$ to W' instead. The only reason why $w_{v,1}$ would be omitted from W' is to destroy the v cycle. However, the same can be accomplished by the woman type $w_{v,2}$, so rationalizability is preserved. Notice that the size of the set W' does not change.
- P2. For all w , if $w \in W \setminus W'$, if $w = w_e$ for some edge $e \in E$, and if $w_{v,2} \in W'$ for one of the two vertices v of the edge e , we add w_e to W' and remove $w_{v,2}$ from W' . The reason why w_e would be omitted from W' is to separate the two cycles corresponding to the two vertices of e . However, by deleting one of these two vertices for the woman type $w_{v,2}$, this accomplishes the same goal. As such, rationalizability is preserved and the size of W' does not change.

After repeated applications of these perturbations, consider the set $\tilde{V}$ of all vertices v such that $w_{v,2} \in W'$. Let us show that this is an independent set of the desired size.

Toward a contradiction, assume that $v, v' \in \tilde{V}$ and $e = (v, v') \in E$. This means that both $w_{v,2}, w_{v',2} \in W'$. If also $w_{v,1}, w_{v',1}, w_e \in W'$, then this violates rationalizability of $\mathcal{M}'$, as there are two distinct connected minimal cycles. As such, at least one of the three types is not in W' .

- If $w_{v,1} \notin W'$, then by the fact that $w_{v,2} \in W'$, this woman type should have been added to W' in exchange for $w_{v,2}$ by the first perturbation (P1) above. A similar reasoning holds if $w_{v',1} \notin W'$.
- If $w_e \notin W'$, then the second perturbation (P2) above requires that both $w_{v,2}$ and $w_{v',2} \notin W'$, a contradiction.

This proves that $\tilde{V}$ is an independent set. Notice that there can be at most $|V| + |E|$ elements in W' that are not of the type $w_{v,2}$. As such, if $|W'| \geq |V| + |E| + k$, it follows that $|\tilde{V}| \geq k$.

MTR-TU Consider the graph $G = (V, E)$ and a number $k \in \mathbb{N}$ as an instance of MIS. We construct an instance of MTR-TU, i.e., a matching (M, W, X) and a number K , in the following way:

- For every vertex $v \in V$, we construct a woman type w_v .
- For every edge $e \in E$ we construct two woman types w_e^1 and w_e^2 .
- For every vertex $v \in V$ and every edge $e \in E$ connected to v , we construct two man types $m_{v,e}^1$ and $m_{v,e}^2$.

To define the matchings among these types we set

$$\begin{aligned}
 X(m_{v,e}^1, w_v) &= 1 && \text{if } e \text{ is adjacent to } v \\
 X(m_{v,e}^2, w_v) &= 1 && \text{if } e \text{ is adjacent to } v \\
 X(m_{v,e}^1, w_e^1) &= 1 && \text{if } e \text{ is adjacent to } v \\
 X(m_{v,e}^2, w_e^2) &= 1 && \text{if } e \text{ is adjacent to } v.
 \end{aligned}$$

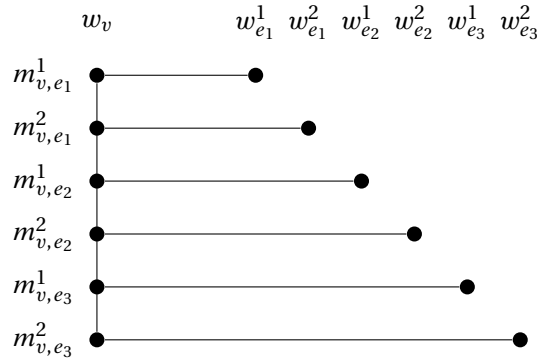


FIGURE 8. Illustration of a configuration of a vertex v with three adjacent edges e_1 , e_2 , and e_3 .

Figure 8 gives the example of the configuration for a vertex v with three adjacent edges, e_1 , e_2 , and e_3 .

To finish the description of the MTR-TU instance, we set the parameter K for the MTR-TU problem equal to $2|E| + k$.

This finishes the instance construction. In total, we have $|V| + 2|E|$ woman types and $2\sum_v d(v) = 4|E|$ man types, where d is the degree of the vertex v . This number is bounded by $6|E| + |V|$, which is polynomial in the inputs.

As an illustrative example of how the instance of MTR-TU may look, consider again the network given in Figure 6 with its four vertices $\{a, b, c, d\}$ and four edges $\{e_1, e_2, e_3, e_4\}$. The MTR-TU instance corresponding to this graph is given in Figure 9.

In order to finish the proof, we need to show that every yes instance of MIS corresponds to a yes instance for the associated MTR-TU problem and vice versa.

Toward this end, let $\{G = (V, E), k\}$ be the instance of the MIS problem and let $\{\mathcal{M} = (M, W, X), K\}$ be the associated instance of the MTR-TU problem. Assume that $\{G = (V, E), k\}$ is a yes instance and let $\tilde{V}$ be an independent set of size $|\tilde{V}| \geq k$. Let us show that there is a woman-type reduction (M, W', X') of size $|W'| \geq K = 2|E| + k$.

For all $v \in V \setminus \tilde{V}$, remove from W the type w_v . Notice that this destroys any cycle containing w_v . This leaves us with $2|E| + |\tilde{V}| \geq 2|E| + k = K$ woman types W' . Now assume that $\mathcal{M}' = (M, W', X')$ is not rationalizable. In this case, there should be a vertex-minimal cycle. Consider a man type $m_{v,e}^i$ that is part of the cycle. As every man type $m_{v,e}^i$ is only matched to w_v and w_e^i , then both $(m_{v,e}^i, w_v)$ and $(m_{v,e}^i, w_e^i)$ must be part of the cycle. As w_e^i is also matched to exactly two man types, i.e., $m_{v,e}^i$ and $m_{v',e}^i$, where $e = (v, v')$, the vertex $(m_{v',e}^i, w_e^i)$ should also be part of the cycle. But then it must be that the vertex $(m_{v',e}^i, w_{v'})$ is also part of the cycle, which means that both vertices v and v' of the edge e are in $\tilde{V}$, a contradiction with the definition of an independent set.

For the reverse, let $\{\mathcal{M} = (M, W, X), K\}$ be a yes instance for MTR-TU and let $\mathcal{M}' = (M, W', X')$ be a woman-type reduction with $|W'| \geq K = 2|E| + k$. We are going to perturb this solution slightly to a new yes instance of MTR-TU in the following way:

- P. For all w , if $w \in W \setminus W'$ with $w = w_e^i$ and if $w_v \in W'$ for some vertex v of edge e , we remove w_v from W' and add w_e^i to W' instead. As any cycle that involves w_e^i must

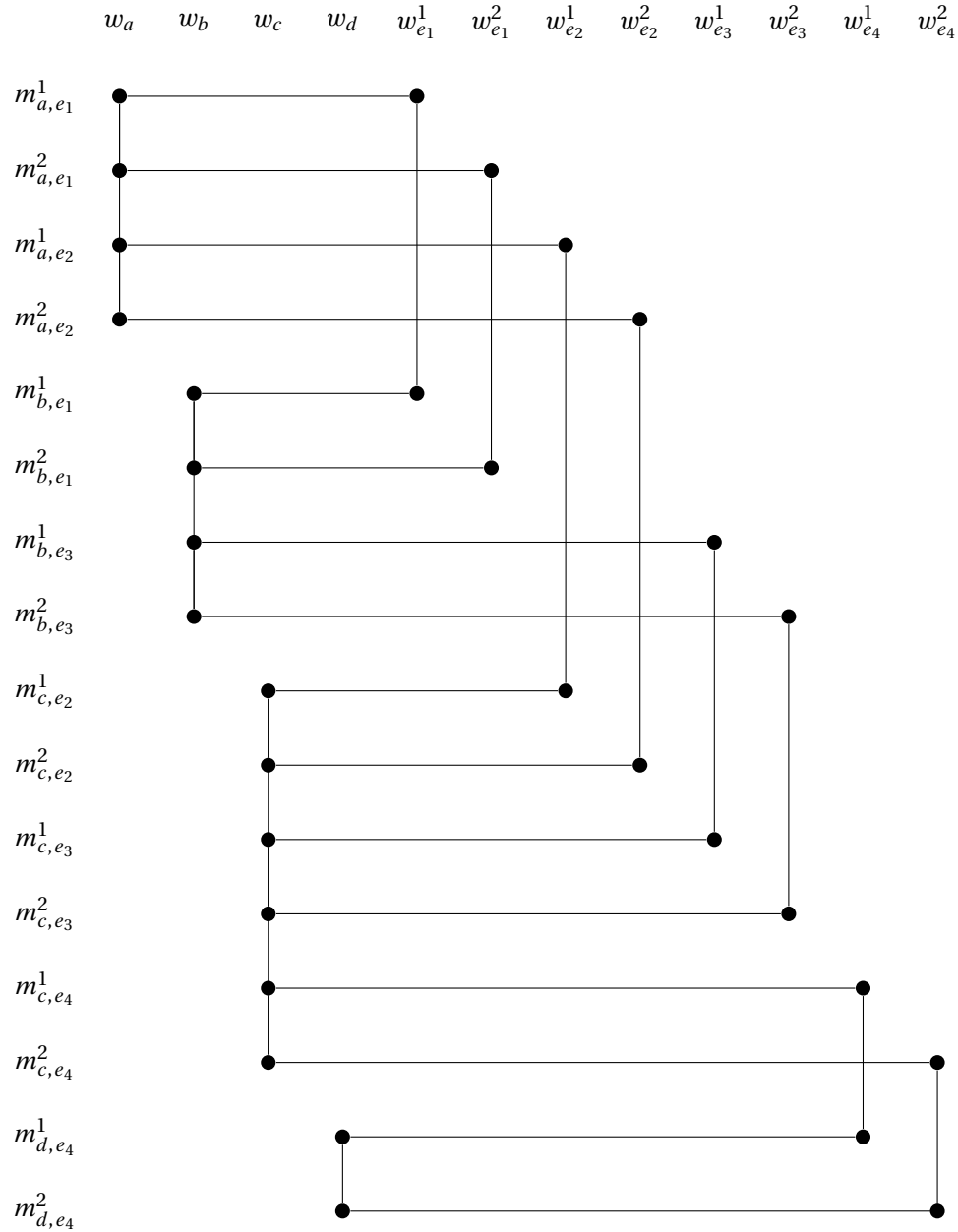


FIGURE 9. MTR-TU instance of the graph in Figure 6.

involve the vertex $(w_e^i, m_{v,e}^i)$ and, therefore, also the vertex $(m_{v,e}^i, w_v)$, it follows that any cycle that is destroyed by removing w_e^i is also destroyed by removing w_v .

After repeated applications of these perturbations, consider the set $\tilde{V}$ of all vertices v such that $w_v \in W'$. Let us show that this is an independent set of the desired size.

Toward a contradiction, assume that $v, v' \in \tilde{V}$ and $e = (v, v') \in E$. This means that both $w_v, w_{v'} \in W'$. If also $w_e^1, w_e^2 \in W'$, then this violates rationalizability of $\mathcal{M}'$, as there is a cycle

$$(m_{v,e}^1, w_v), (m_{v,e}^1, w_e^1), (m_{v',e}^1, w_e^1), (m_{v',e}^1, w_{v'}), (m_{v',e}^2, w_{v'}), \dots, \\ (m_{v',e}^2, w_e^2), (m_{v,e}^2, w_e^2), (m_{v,e}^2, w_v), (m_{v,e}^1, w_v).$$

As such, at least one of the types w_e^1 or w_e^2 is not in W' . However, then by applying the perturbation P , either w_v or $w_{v'}$ should also have been removed. This proves that $\tilde{V}$ is an independent set. Notice that there are at most $2|E|$ elements in W' of type w_e^i . A such, if $|W'| \geq 2|E| + k$, it follows that $\tilde{V} \geq k$.

APPENDIX B: SUMMARY STATISTICS

TABLE 2. Summary statistics for marriage market illustration.

Variable	Women		Men	
	Mean	St. Dev.	Mean	St. Dev.
Education	1.87	0.566	2.01	0.571
Height	169.35	6.406	182.34	7.199
BMI	23.44	3.831	24.53	2.944
Age	32.78	4.841	35.52	6.009

Note: Education takes values 1, 2, or 3 (see main text), height is measured in centimeters, BMI is defined as weight (in kilograms) divided by height (in meters) squared, and age is in years.

APPENDIX C: A CHARACTERIZATION OF THE L_1 DISTANCE FUNCTION

Let $\overline{M}$ be a finite universal set of men types and let $\overline{W}$ be a finite universal set of women types. For nonempty $M \subseteq \overline{M}$ and $W \subseteq \overline{W}$, let us write the distance between two matchings $\mathcal{M} = (M, W, X)$ and $\mathcal{M}' = (M, W, \tilde{X})$ by $d(\mathcal{M}, \mathcal{M}')$. The distance is only defined if the matchings have the same set of men and women types.

For ease of notation, we introduce a function $k(M, W, X, X')$ such that for $\mathcal{M} = (M, W, X)$ and $\mathcal{M}' = (M, W, X')$,

$$k(M, W, X, X') \equiv d(\mathcal{M}, \mathcal{M}').$$

We impose several assumptions (axioms) on the function d (or, equivalently, k). First of all, we assume that d is not equal to the zero function and that it is symmetric.

ASSUMPTION 1 (Nontriviality). *There are matchings $\mathcal{M} = (M, W, X)$, $\mathcal{M}' = (M, W, X')$ such that*

$$k(M, W, X, X') > 0.$$

ASSUMPTION 2 (Symmetry). *For all matchings $\mathcal{M} = (M, W, X)$ and $\mathcal{M}' = (M, W, X')$,*

$$k(M, W, X, X') = k(M, W, X', X).$$

Next, we impose two anonymity conditions. The first requires that the distance should not distinguish between men and women types.

ASSUMPTION 3 (Anonymity I). *For all matchings $\mathcal{M} = (M, W, X)$ and $\mathcal{M}' = (M, W, X')$,*

$$k(M, W, X, X') = k(W, M, X^T, (X')^T),$$

where X^T and $(X')^T$ are the transpose of X and X' , respectively.

The second anonymity assumption requires that our distance measure does not depend on the labels of the types.

ASSUMPTION 4 (Anonymity II). *For a permutation $\sigma : M \rightarrow M$, let us denote by M_σ the set $\{m_{\sigma(1)}, \dots, m_{\sigma(|M|)}\}$. Then, for all matchings $\mathcal{M} = (M, W, X)$,*

$$k(M, W, X) = k(M_\sigma, W, X).$$

The following assumption requires that the distance is additively separable over subgroups of types.

ASSUMPTION 5 (Separability). *Let $M = M_1 \cup M_2$ and $M_1 \cap M_2 = \emptyset$. Let $X_1 = (X_{m,w})_{m \in M_1, w \in W}$ and $X_2 = (X_{m,w})_{m \in M_2, w \in W}$, and similar for X'_1 and X'_2 . Then*

$$k(M, W, X, X') = k(M_1, W, X_1, X'_1) + k(M_2, W, X_2, X'_2).$$

Define a simple matching to be a matching with only one man type and one woman type. The following assumption requires that for these simple matchings, multiplying the number of matchings by some strictly positive number scales the distance by the same number.

ASSUMPTION 6 (Homogeneity of Degree 1). *Let $\alpha \in \mathbb{N}$ ($\alpha > 0$). Then for all simple matchings $\mathcal{M} = (\{m\}, \{w\}, x)$ and $\mathcal{M}' = (\{m\}, \{w\}, x')$,*

$$k(\{m\}, \{w\}, \alpha x, \alpha x') = \alpha k(\{m\}, \{w\}, x, x').$$

Finally, we assume that for two simple matchings, subtracting the same number of matches from both does not change the distance between the two.

ASSUMPTION 7 (Translation Invariance). *Let $a \in \mathbb{N}$, $x, x' \in \mathbb{N}$, and $a < \min\{x, x'\}$. Then, for all simple matchings $\mathcal{M} = (\{m\}, \{w\}, x)$,*

$$k(\{m\}, \{w\}, x, x') = k(\{m\}, \{w\}, x - a, x' - a).$$

We then have the following result.

PROPOSITION 12. *Assumptions 1–7 are satisfied if and only if there exists a $\kappa > 0$, such that for all matchings $\mathcal{M} = (M, W, X)$ and $\mathcal{M}' = (M, W, \tilde{X})$,*

$$d(\mathcal{M}, \mathcal{M}') = \kappa \sum_{m,w} |X(m, w) - \tilde{X}(m, w)|.$$

Moreover, all assumptions are independent.

PROOF. It is easy to verify that the specification in the proposition satisfies all assumptions. For the reverse, notice that by repeated application of Separability, and Anonymity I,

$$d(\mathcal{M}, \mathcal{M}') = \sum_{m \in M, w \in W} k(\{m\}, \{w\}, X(m, w), \tilde{X}(m, w)).$$

By Anonymity I and II, the values $k(\{m\}, \{w\}, X(m, w), \tilde{X}(m, w))$ are independent of the labels m and w , and, therefore, only depend on the values of $X(m, w)$ and $\tilde{X}(m, w)$. As such, for all $x, \tilde{x} \in \mathbb{N} \cup \{0\}$, we can define the function

$$\delta(x, x') \equiv k(\{m\}, \{w\}, x, x').$$

If $x = x' = 0$, then by Homogeneity of Degree 1,

$$\delta(0, 0) = \alpha \delta(0, 0)$$

for all $\alpha \in \mathbb{N}$ ($\alpha > 0$), so $\delta(0, 0) = 0$. Next, by Homogeneity of Degree 1, Translation Invariance, and Symmetry, we obtain

$$\begin{aligned} \delta(x, x') &= \delta(x - \min\{x, x'\}, \tilde{x} - \min\{x, x'\}) \\ &= \delta(0, |x - x'|) \\ &= |x - x'| \delta(0, 1). \end{aligned}$$

If we let $\kappa = \delta(0, 1)$, this gives

$$d(\mathcal{M}, \mathcal{M}') = \kappa \sum_{m,w} |X(m, w) - \tilde{X}(m, w)|.$$

By nontriviality, we have $\kappa > 0$.

To show independence, we give, for each property, a distance that satisfies all but that single property.

- Nontriviality:

$$d(\mathcal{M}, \mathcal{M}') = 0.$$

- Symmetry:

$$d(\mathcal{M}, \mathcal{M}') = \sum_{m \in M, w \in W} (X(m, w) - \tilde{X}(m, w)).$$

- Anonymity I:

$$d(\mathcal{M}, \mathcal{M}') = \sum_{m \in M} \left| \sum_{w \in W} X(m, w) - \sum_{w \in W} X'(m, w) \right|.$$

- Anonymity II:

$$d(\mathcal{M}, \mathcal{M}') = \sum_{m \in M, w \in W} r_m r_w |X(m, w) - X'(m, w)|,$$

where r_m and r_w are the order of m and w in the universal type sets $\overline{M}$ and $\overline{W}$ respectively (for some arbitrary ranking).

- Separability:

$$d(\mathcal{M}, \mathcal{M}') = \left| \prod_{m \in M, w \in W} X(m, w) - \prod_{m \in M, w \in W} X'(m, w) \right|.$$

- Homogeneity:

$$d(\mathcal{M}, \mathcal{M}') = \sum_{m \in M, w \in W} (X(m, w) - X'(m, w))^2.$$

- Translation invariance:

$$d(\mathcal{M}, \mathcal{M}') = \sum_{m \in M, w \in W} ((X(m, w))^2 - (X'(m, w))^2)^{1/2}. \quad \square$$

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