

Supplement to “Observational learning in large anonymous games”: Omitted proofs

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S.1. PROOF OF LEMMA 6

Take a limit point $x = (x_0, x_1)$ with $v_0(x_0) > 0$ and $v_1(x_1) < 0$. In the limit, agents want their action to go against the state of the world. Now the simple strategy $\tilde{\sigma}^T$ is

$$\tilde{\sigma}^T(\tilde{\xi}, s) = \begin{cases} 1 & \text{if } \tilde{\xi} = 1 \text{ and } l(s) \leq \underline{k}^T \equiv \frac{v_0(E_{\sigma^T}[X_0])}{-v_1(E_{\sigma^T}[X_1])} \frac{\mathbf{P}_{\sigma^T}(\tilde{\xi} = 1 \mid \theta = 0)}{\mathbf{P}_{\sigma^T}(\tilde{\xi} = 1 \mid \theta = 1)} \\ 1 & \text{if } \tilde{\xi} = 0 \text{ and } l(s) \leq \bar{k}^T \equiv \frac{v_0(E_{\sigma^T}[X_0])}{-v_1(E_{\sigma^T}[X_1])} \frac{\mathbf{P}_{\sigma^T}(\tilde{\xi} = 0 \mid \theta = 0)}{\mathbf{P}_{\sigma^T}(\tilde{\xi} = 0 \mid \theta = 1)} \\ 0 & \text{otherwise.} \end{cases}$$

Given this simple strategy, the approximate improvement is given by

$$\begin{aligned} \Delta^T &= \frac{1}{2} \sum_{\theta \in \{0,1\}} [\mathbf{P}_{\tilde{\sigma}^T}(a_i = 1 \mid \theta) - E_{\sigma^T}[X_\theta]] \cdot v_\theta(E_{\sigma^T}[X_\theta]) \\ &= \frac{1}{2} \sum_{\theta \in \{0,1\}} [\varepsilon + (1 - 2\varepsilon)[\pi_\theta^T G_\theta(\underline{k}^T) + (1 - \pi_\theta^T) G_\theta(\bar{k}^T)] - E_{\sigma^T}[X_\theta] \cdot v_\theta(E_{\sigma^T}[X_\theta]) \\ &= \frac{1}{2} \sum_{\theta \in \{0,1\}} v_\theta(E_{\sigma^T}[X_\theta]) [\varepsilon + (1 - 2\varepsilon)[\pi_\theta^T [G_\theta(\underline{k}^T) - 1] + (1 - \pi_\theta^T) G_\theta(\bar{k}^T)]] \\ &\quad + v_\theta(E_{\sigma^T}[X_\theta]) [(1 - 2\varepsilon)\pi_\theta - E_{\sigma^T}[X_\theta]] \\ &= \frac{1}{2} \sum_{\theta \in \{0,1\}} v_\theta(E_{\sigma^T}[X_\theta]) [(1 - 2\pi_\theta^T)\varepsilon + (1 - 2\varepsilon)[\pi_\theta^T [G_\theta(\underline{k}^T) - 1] + (1 - \pi_\theta^T) G_\theta(\bar{k}^T)]] \\ &\quad + \frac{1}{2} \sum_{\theta \in \{0,1\}} v_\theta(E_{\sigma^T}[X_\theta]) [\pi_\theta - E_{\sigma^T}[X_\theta]]. \end{aligned}$$

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Thus,

$$\begin{aligned}
\Delta^T &= \frac{1}{2}[(1 - 2\pi_0^T)\varepsilon + (1 - 2\varepsilon)[- \pi_0^T[1 - G_0(\underline{k}^T)] + (1 - \pi_0^T)G_0(\bar{k}^T)]] \cdot v_0(E_{\sigma^T}[X_0]) \\
&\quad + \frac{1}{2}[(1 - 2\pi_1^T)\varepsilon + (1 - 2\varepsilon)[- \pi_1^T[1 - G_1(\underline{k}^T)] + (1 - \pi_1^T)G_1(\bar{k}^T)]] \cdot v_1(E_{\sigma^T}[X_1]) \\
&\quad + \frac{1}{2} \sum_{\theta \in \{0,1\}} v_\theta(E_{\sigma^T}[X_\theta])[\pi_\theta - E_{\sigma^T}[X_\theta]] \\
&= \frac{1}{2}[(1 - 2\varepsilon)(1 - \pi_0^T) \left[G_0(\bar{k}^T) - \frac{-v_1(E_{\sigma^T}[X_1])}{v_0(E_{\sigma^T}[X_0])} \frac{(1 - \pi_1^T)}{(1 - \pi_0^T)} G_1(\bar{k}^T) \right]] \cdot v_0(E_{\sigma^T}[X_0]) \\
&\quad + \frac{1}{2}[(1 - 2\varepsilon)\pi_1^T \left[\frac{v_0(E_{\sigma^T}[X_0])}{-v_1(E_{\sigma^T}[X_1])} \frac{\pi_0^T}{\pi_1^T} [1 - G_0(\underline{k}^T)] - [1 - G_1(\underline{k}^T)] \right]] \cdot v_1(E_{\sigma^T}[X_1]) \\
&\quad + \frac{1}{2}(1 - 2\pi_0^T)\varepsilon \cdot v_0(E_{\sigma^T}[X_0]) + \frac{1}{2}(1 - 2\pi_1^T)\varepsilon \cdot v_1(E_{\sigma^T}[X_1]) \\
&\quad + \frac{1}{2} \sum_{\theta \in \{0,1\}} v_\theta(E_{\sigma^T}[X_\theta])[\pi_\theta - E_{\sigma^T}[X_\theta]] \\
&= \frac{1}{2}[(1 - 2\pi_0^T)\varepsilon + (1 - 2\varepsilon)(1 - \pi_0^T)[G_0(\bar{k}^T) - (\bar{k}^T)^{-1}G_1(\bar{k}^T)]] \cdot v_0(E_{\sigma^T}[X_0]) \\
&\quad + \frac{1}{2}[(2\pi_1^T - 1)\varepsilon + (1 - 2\varepsilon)\pi_1^T[[1 - G_1(\underline{k}^T)] - \underline{k}^T[1 - G_0(\underline{k}^T)]]] \cdot (-v_1(E_{\sigma^T}[X_1])) \\
&\quad + \frac{1}{2} \sum_{\theta \in \{0,1\}} v_\theta(E_{\sigma^T}[X_\theta])[\pi_\theta - E_{\sigma^T}[X_\theta]].
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{T \rightarrow \infty} \Delta^T &= \frac{1}{2}[(1 - 2x_0)\varepsilon + (1 - 2\varepsilon)(1 - x_0)[G_0(\bar{k}) - (\bar{k})^{-1}G_1(\bar{k})]] \cdot v_0(x_0) \\
&\quad + \frac{1}{2}[(2x_1 - 1)\varepsilon + (1 - 2\varepsilon)x_1[[1 - G_1(\underline{k})] - \underline{k}[1 - G_0(\underline{k})]]] \cdot (-v_1(x_1)).
\end{aligned}$$

Again, Corollary 2 leads directly to

$$\begin{aligned}
&[(1 - 2\varepsilon)(1 - x_0)[G_0(\bar{k}) - (\bar{k})^{-1}G_1(\bar{k})] - \varepsilon(2x_0 - 1)] \cdot v_0(x_0) \\
&+ [(1 - 2\varepsilon)x_1[[1 - G_1(\underline{k})] - \underline{k}[1 - G_0(\underline{k})]] - \varepsilon(1 - 2x_1)] \cdot (-v_1(x_1)) \leq 0.
\end{aligned}$$

S.2. PROOF OF LEMMA 7

Let $\widetilde{\text{NE}}_\delta = \{x \in [0, 1]^2 : d(x, \text{NE}_{(\underline{L}, \bar{I})}) \leq \delta\}$ be the set of all points that are δ -close to elements of $\text{NE}_{(\underline{L}, \bar{I})}$ and let L^ε denote the set of limit points in a game with mistake probability $\varepsilon > 0$. I show first the following lemma, which is analogous to Lemma 11 in the main paper.

LEMMA 11' (Limit set approaches $\text{NE}_{(\underline{L}, \bar{I})}$). *For any $\delta > 0$, $\exists \tilde{\varepsilon} > 0$, $L^\varepsilon \subseteq \widetilde{\text{NE}}_\delta \forall \varepsilon < \tilde{\varepsilon}$.*

PROOF. The proof is by contradiction. Assume that there exists (i) a sequence of mistake probabilities $\{\varepsilon^n\}_{n=1}^\infty$ with $\lim_{n \rightarrow \infty} \varepsilon^n = 0$ and (ii) an associated sequence $\{x^n\}_{n=1}^\infty$ with $x^n \in L^{\varepsilon^n}$ for all n , but (iii) $x^n \notin \widetilde{\text{NE}}_\delta$ for all n . Since $x^n \in [0, 1]^2$ for all n , this sequence has a convergent subsequence $\{x^{n_m}\}_{m=1}^\infty$ with $\lim_{m \rightarrow \infty} x^{n_m} = \bar{x} = (\bar{x}_0, \bar{x}_1)$. If $v_0(\bar{x}_0) = v_1(\bar{x}_1) = 0$, then $\bar{x} \in \text{NE}$, so for m large enough, $x^{n_m} \in \widetilde{\text{NE}}_\delta$. Then it must be the case that $v_\theta(\bar{x}_\theta) \neq 0$ for some θ .

Assume that $v_1(\bar{x}_1) > 0$. Pick $\tilde{m}$ large enough so that $v_1(x_1^{n_m}) > 0$ for all $m > \tilde{m}$. For all m with $v_0(x_0^{n_m}) \geq 0$, Lemma 4 implies that $x^{n_m} = (1 - \varepsilon^{n_m}, 1 - \varepsilon^{n_m})$. So if $v_0(x_0^{n_m}) \geq 0$ infinitely often, then $\bar{x} = (1, 1)$. As a result, $\bar{x} \in \text{NE}$, so for m large enough, $x^{n_m} \in \widetilde{\text{NE}}_\delta$.

Take next all m with $v_0(x_0^{n_m}) < 0$. By Lemma 5, (3) must hold:

$$\begin{aligned} & \frac{-v_0(x_0^{n_m})}{2} \left[\overbrace{(1 - 2\varepsilon^{n_m})}^{\rightarrow 1} x_0^{n_m} \left[\overbrace{G_0(\underline{k}^{n_m}) - (\underline{k}^{n_m})^{-1} G_1(\underline{k}^{n_m})}^{\geq 0} \right] - \overbrace{\varepsilon(1 - 2x_0)}^{\rightarrow 0} \right] \\ & + \frac{v_1(x_1^{n_m})}{2} \left[\overbrace{(1 - 2\varepsilon^{n_m})}^{\rightarrow 1} (1 - x_1^{n_m}) \left[\overbrace{[1 - G_1(\bar{k}^{n_m})] - \bar{k}^{n_m} [1 - G_0(\bar{k}^{n_m})]}^{\geq 0} \right] \right] \\ & - \underbrace{\varepsilon^{n_m} (2x_1^{n_m} - 1)}_{\rightarrow 0} \leq 0. \end{aligned} \tag{S.1}$$

Proposition 3 guarantees both that $[1 - G_1(\bar{k}^{n_m})] - \bar{k}^{n_m} [1 - G_0(\bar{k}^{n_m})] \geq 0$ and that $[G_0(\underline{k}^{n_m}) - (\underline{k}^{n_m})^{-1} G_1(\underline{k}^{n_m})] \geq 0$. Then, as (S.1) shows, when $\varepsilon^{n_m} \rightarrow 0$, only nonnegative terms may remain. Assume that $\bar{k} = -[v_0(\bar{x}_0)(1 - \bar{x}_0)]/[v_1(\bar{x}_1)(1 - \bar{x}_1)] < \bar{l}$. Then, for ε small enough, $\bar{k}^{n_m} < \bar{l}$. Proposition 3 implies that

$$\lim_{m \rightarrow \infty} [[1 - G_1(\bar{k}^{n_m})] - \bar{k}^{n_m} [1 - G_0(\bar{k}^{n_m})]] > 0.$$

To summarize, whenever $\bar{k} < \bar{l}$, (S.1) is not satisfied for small enough ε^{n_m} . It must be the case then that $\bar{k} \geq \bar{l}$. Similarly, if $\underline{k} > \underline{l}$, then

$$\lim_{m \rightarrow \infty} [G_0(\underline{k}^{n_m}) - (\underline{k}^{n_m})^{-1} G_1(\underline{k}^{n_m})] > 0$$

for small enough ε^{n_m} . It must be the case then that $\underline{k} \leq \underline{l}$.

Analogous arguments (using also Lemma 6) lead to the same result for the case with $v_1(\bar{x}_1) < 0$. As a result, $\bar{x} \in \text{NE}_{(\underline{l}, \bar{l})}$, so for m large enough, $x^{n_m} \in \widetilde{\text{NE}}_\delta$.

The rest of the proof is identical to the proof of Proposition 2 in the paper. $\square$

S.3. EXAMPLE 4: STANDARD OBSERVATIONAL LEARNING WITH MISTAKES

This corresponds to Example 4 in the paper. Utility is given by $u(1, X, 1) = u(0, X, 0) = 1$ and $u(1, X, 0) = u(0, X, 1) = 0$. Each agent observes his immediate predecessor: $M = 1$. The signal structure is described by $\nu_1[(0, s)] = s^2$ and $\nu_0[(0, s)] = 2s - s^2$ with $s \in (0, 1)$.

PROOF OF EXAMPLE 4. Let $\pi \equiv \Pr(\xi = 1 \mid \theta = 1)$. An agent who observes $\xi = 1$ chooses action 1 if and only if $\frac{\pi}{1-\pi} \frac{s}{1-s} \geq 1 \Leftrightarrow s \geq 1 - \pi$. Similarly, an agent who observes $\xi =$

0 chooses action 1 if and only if $\frac{1-\pi}{\pi} \frac{s}{1-s} \geq 1 \Leftrightarrow s \geq \pi$. As a result, the likelihood that somebody who observes a sample (that is, not agent 1) will choose the right action is given by

$$\begin{aligned}
 \Pr(a_i = 1 \mid \theta = 1, Q(i) \neq 1) &= \frac{1}{T-1} \sum_{t=2}^T \Pr(a_t = 1 \mid \theta = 1) \\
 &= \varepsilon + (1-2\varepsilon) [\pi \Pr(s \geq 1-\pi) + (1-\pi) \Pr(s \geq \pi)] \\
 &= \varepsilon + (1-2\varepsilon) [\pi [1 - (1-\pi)^2] + (1-\pi) [1 - \pi^2]] \\
 &= \varepsilon + (1-2\varepsilon) [\pi - \pi(1 + \pi^2 - 2\pi) + 1 - \pi - \pi^2 + \pi^3] \\
 &= \varepsilon + (1-2\varepsilon) [\pi - \pi - \pi^3 + 2\pi^2 + 1 - \pi - \pi^2 + \pi^3] \\
 &= \varepsilon + (1-2\varepsilon)(1 - \pi + \pi^2).
 \end{aligned}$$

Reordering yields

$$\Pr(a_1 = 1 \mid \theta = 1) + \sum_{t=2}^T \Pr(a_t = 1 \mid \theta = 1) = \sum_{t=1}^{T-1} \Pr(a_t = 1 \mid \theta = 1) + \Pr(a_T = 1 \mid \theta = 1).$$

Then

$$\begin{aligned}
 \varepsilon + (1-2\varepsilon)(1 - \pi + \pi^2) - \pi - \frac{\Pr(a_T = 1 \mid \theta = 1) - \Pr(a_1 = 1 \mid \theta = 1)}{T-1} &= 0 \\
 \varepsilon + (1-2\varepsilon)(1 - \pi + \pi^2) - \pi - \Delta &= 0 \\
 (1-2\varepsilon)\pi^2 - 2(1-\varepsilon)\pi + 1 - \varepsilon - \Delta &= 0,
 \end{aligned}$$

where I define $\Delta \equiv \frac{\Pr(a_T=1|\theta=1)-\Pr(a_1=1|\theta=1)}{T-1}$. Then

$$\begin{aligned}
 \pi &= \frac{2(1-\varepsilon) \pm \sqrt{4(1-\varepsilon)^2 - 4(1-2\varepsilon)(1-\varepsilon-\Delta)}}{2(1-2\varepsilon)} \\
 &= \frac{1-\varepsilon - \sqrt{(1-\varepsilon)^2 - (1-2\varepsilon)(1-\varepsilon-\Delta)}}{1-2\varepsilon}.
 \end{aligned}$$

Note that $\lim_{T \rightarrow \infty} \Delta = 0$. Then

$$\begin{aligned}
 \pi &\rightarrow \frac{1-\varepsilon - \sqrt{(1-\varepsilon)^2 - (1-2\varepsilon)(1-\varepsilon)}}{1-2\varepsilon} \\
 &= \frac{1-\varepsilon}{1-2\varepsilon} \left(1 - \sqrt{1 - \frac{1-2\varepsilon}{1-\varepsilon}} \right) = \frac{1-\varepsilon}{1-2\varepsilon} \left(1 - \sqrt{\frac{\varepsilon}{1-\varepsilon}} \right).
 \end{aligned}$$

Also, as $T \rightarrow \infty$, $\pi - \Pr(a_i = 1 \mid \theta) \rightarrow 0$. Then $x_1 = \lim_{T \rightarrow \infty} \Pr(a_i = 1 \mid \theta) = \frac{1-\varepsilon}{1-2\varepsilon} (1 - \sqrt{\frac{\varepsilon}{1-\varepsilon}})$. $\square$

S.4. EXAMPLE 8: MULTIPLE EQUILIBRIA IN A COORDINATION GAME

PROOF OF EXAMPLE 8. Consider a sequence of symmetric strategy profiles $\{\sigma^T(s, \xi)\}$, where $\sigma^T(s, \xi) = \sigma(s, \xi)$ does not change with T and is given by

$$\sigma(s, \xi) = \begin{cases} 1 & \text{if } s = 1 \\ 0 & \text{if } s = 0 \\ \xi & \text{if } s = 1/2. \end{cases}$$

Let $\pi \equiv \Pr(\xi = 1 \mid \theta = 1)$. Under $\sigma(s, \xi)$, the likelihood that somebody who observes a sample (that is, not agent 1) chooses action 1 is given by

$$\begin{aligned} \Pr(a_i = 1 \mid \theta = 1, Q(i) \neq 1) &= \frac{1}{T-1} \sum_{t=2}^T \Pr(a_t = 1 \mid \theta = 1) \\ &= \varepsilon + (1 - 2\varepsilon)[\Pr(s = 1) + \Pr(s = 1/2)\pi] \\ &= \varepsilon + (1 - 2\varepsilon)[(1 - \gamma)/100 + 99/100\pi]. \end{aligned}$$

Reordering yields

$$\Pr(a_1 = 1 \mid \theta = 1) + \sum_{t=2}^T \Pr(a_t = 1 \mid \theta = 1) = \sum_{t=1}^{T-1} \Pr(a_t = 1 \mid \theta = 1) + \Pr(a_T = 1 \mid \theta = 1).$$

Then

$$\begin{aligned} &\frac{\sum_{t=2}^T \Pr(a_t = 1 \mid \theta = 1)}{T-1} - \frac{\sum_{t=1}^{T-1} \Pr(a_t = 1 \mid \theta = 1)}{T-1} \\ &= \frac{\Pr(a_T = 1 \mid \theta = 1) - \Pr(a_1 = 1 \mid \theta = 1)}{T-1}, \end{aligned}$$

so

$$\begin{aligned} \Pr(a_i = 1 \mid \theta = 1, Q(i) \neq 1) - \pi &= \frac{\Pr(a_T = 1 \mid \theta = 1) - \Pr(a_1 = 1 \mid \theta = 1)}{T-1} \\ \varepsilon + (1 - 2\varepsilon)[(1 - \gamma)/100 + 99/100\pi] - \pi &= \Delta. \end{aligned}$$

Then

$$\begin{aligned} \varepsilon - 2\varepsilon[(1 - \gamma)/100 + 99/100\pi] + (1 - \gamma)/100 - 1/100\pi &= \Delta \\ \varepsilon - 2\varepsilon(1 - \gamma)/100 - \varepsilon 198/100\pi + (1 - \gamma)/100 - 1/100\pi &= \Delta \\ +(1 - \gamma)/100 + [1 - (1 - \gamma)/50]\varepsilon - (1/100 + 198/100\varepsilon)\pi &= \Delta \\ +(1 - \gamma) + [100 - 2(1 - \gamma)]\varepsilon - (1 + 198\varepsilon)\pi &= 100\Delta. \end{aligned}$$

Then

$$\pi = \frac{(1 - \gamma) + [100 - 2(1 - \gamma)]\varepsilon - 100\Delta}{1 + 198\varepsilon}.$$

Proposition 1 guarantees that as the number of agents grows large, the average action is close to its expectation. For low enough ε and large enough T , approximately $X_0|\sigma \xrightarrow{P} \gamma$ and $X_1|\sigma \xrightarrow{P} 1 - \gamma$. Then

$$\frac{\Pr(\theta = 1 | \xi = 1)}{\Pr(\theta = 0 | \xi = 1)} \approx \frac{1 - \gamma}{\gamma}.$$

So the sample is informative about the state of the world. To sum up, there is ε small and T large such that σ is indeed an equilibrium. $\square$

S.5. PROVING LEMMA 12

I illustrate first the effect of different values of $\gamma > 1$ on sampling probabilities. Figure S.1 presents an agent in position 21. The black line shows the probability of observing a predecessor in position $\tau < 21$ when $\gamma = 8$. With probability higher than 0.998, the agent observes one of his three immediate predecessors. The distribution becomes flatter as γ decreases. The red line shows the distribution when $\gamma = 1.05$. In this case, the agent in position 21 observes his immediate predecessor twice as often as he observes the first agent in the sequence. As $\gamma \rightarrow 1$, sampling approaches uniform random sampling. Instead, as $\gamma \rightarrow \infty$, sampling approaches observing the immediate predecessor.

Next, I present the proof of Lemma 12.

PROOF OF LEMMA 12. A strategy σ_i induces $\rho_\theta(\xi) = \mathbf{P}_{\sigma_i}(a_i | \theta, \xi)$. For the rest of this section, I fix the state of the world θ and drop its index. Then a strategy σ_i induces a vector $(\rho(\emptyset), \rho(0), \rho(1))$. Because of mistakes, $\varepsilon < \rho(\xi) < 1 - \varepsilon$ for all $\xi \in \{0, 1, \emptyset\}$.

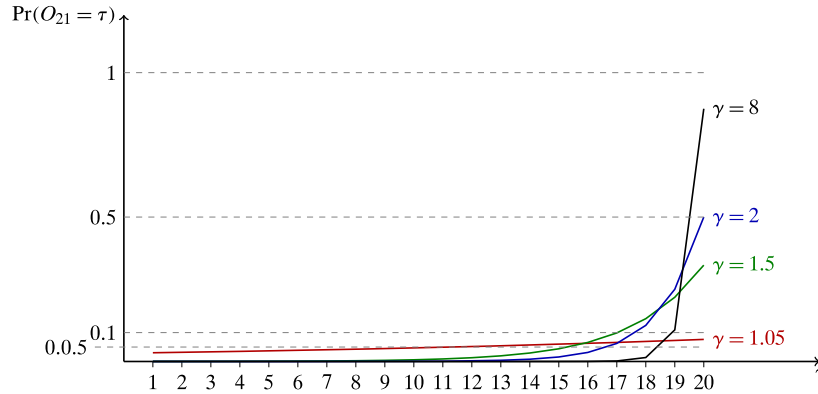


FIGURE S.1. Probabilities of different predecessors being observed: geometric sampling.

Assume first that $\gamma > 1$. The first agent in the sequence chooses action 1 with probability $\rho(\emptyset)$. For $t \geq 2$,

$$\begin{aligned}
\mathbf{P}_\sigma(a_t = 1) &= \Pr(\xi_t = 0) \Pr(a_t = 1 \mid \xi_t = 0) + \Pr(\xi_t = 1) \Pr(a_t = 1 \mid \xi_t = 1) \\
&= \Pr(\xi_t = 0) \rho(0) + \Pr(\xi_t = 1) \rho(1) \\
&= [1 - \Pr(\xi_t = 1)] \rho(0) + \Pr(\xi_t = 1) \rho(1) \\
&= \rho(0) + [\rho(1) - \rho(0)] \Pr(\xi_t = 1) \\
&= \rho(0) + [\rho(1) - \rho(0)] \sum_{\tau < t} \Pr(O_t = \tau) \mathbb{1}\{a_\tau = 1\} \\
&= \rho(0) + [\rho(1) - \rho(0)] \sum_{\tau=1}^{t-1} \frac{\gamma-1}{\gamma} \frac{\gamma^\tau}{\gamma^{t-1}-1} a_\tau.
\end{aligned}$$

Define the weighted sum of the past history by $p_t \equiv \sum_{\tau=1}^{t-1} \frac{\gamma-1}{\gamma} \frac{\gamma^\tau}{\gamma^{t-1}-1} a_\tau$ for $t \geq 2$. This concept plays a key role in the model:

$$\mathbf{P}_\sigma(a_t = 1) = \rho(0) + [\rho(1) - \rho(0)] p_t.$$

This weighted sum has a recursive nature:

$$\begin{aligned}
p_{t+1} &= \sum_{\tau=1}^t \frac{\gamma-1}{\gamma} \frac{\gamma^\tau}{\gamma^t-1} a_\tau = \frac{\gamma^{t-1}-1}{\gamma^t-1} \left[\sum_{\tau=1}^{t-1} \frac{\gamma-1}{\gamma} \frac{\gamma^\tau}{\gamma^{t-1}-1} a_\tau \right] + \frac{\gamma-1}{\gamma} \frac{\gamma^t}{\gamma^t-1} a_t \\
&= \frac{\gamma^{t-1}-1}{\gamma^t-1} p_t + \frac{\gamma^t - \gamma^{t-1}}{\gamma^t-1} a_t.
\end{aligned}$$

In expectation,

$$\begin{aligned}
E[p_{t+1} \mid I_t] &= \frac{\gamma^{t-1}-1}{\gamma^t-1} E[p_t \mid I] + \frac{\gamma^t - \gamma^{t-1}}{\gamma^t-1} E[a_t \mid I] \\
&= \frac{\gamma^{t-1}-1}{\gamma^t-1} E[p_t \mid I] + \frac{\gamma^t - \gamma^{t-1}}{\gamma^t-1} [\rho(0) + [\rho(1) - \rho(0)] E[p_t \mid I]] \\
&= \frac{\gamma^t - 1 + \gamma^{t-1} - \gamma^t}{\gamma^t-1} E[p_t \mid I] + \frac{\gamma^t - \gamma^{t-1}}{\gamma^t-1} [\rho(0) + [\rho(1) - \rho(0)] E[p_t \mid I]] \\
&= E[p_t \mid I] + \frac{\gamma^{t-1} - \gamma^t}{\gamma^t-1} E[p_t \mid I] + \frac{\gamma^t - \gamma^{t-1}}{\gamma^t-1} [\rho(0) + [\rho(1) - \rho(0)] E[p_t \mid I]] \\
&= E[p_t \mid I] + \frac{\gamma^t - \gamma^{t-1}}{\gamma^t-1} [\rho(0) - [1 + \rho(0) - \rho(1)] E[p_t \mid I]] \\
&= E[p_t \mid I] + \frac{\gamma^t - \gamma^{t-1}}{\gamma^t-1} [1 + \rho(0) - \rho(1)] [\rho^* - E[p_t \mid I]].
\end{aligned}$$

Let $\rho^* \equiv \frac{\rho(0)}{1+\rho(0)-\rho(1)}$.¹ Then

$$\begin{aligned}
 E[p_{t+1} | I] - \rho^* &= E[p_t | I] - \rho^* - \frac{\gamma^t - \gamma^{t-1}}{\gamma^t - 1} [1 + \rho(0) - \rho(1)] [E[p_t | I] - \rho^*] \\
 &= \left[1 - \frac{\gamma^t - \gamma^{t-1}}{\gamma^t - 1} [1 + \rho(0) - \rho(1)] \right] [E[p_t | I] - \rho^*] \\
 &= \left[1 - \underbrace{\frac{\gamma - 1}{\gamma}}_{(*)} \underbrace{\frac{\gamma^t}{\gamma^t - 1} [1 + \rho(0) - \rho(1)]}_{(**)} \right] [E[p_t | I] - \rho^*]. \tag{S.2}
 \end{aligned}$$

I next provide bounds for the terms $(*)$ and $(**)$ in (S.2):

$$\begin{aligned}
 2\varepsilon &\leq 1 + \rho(0) - \rho(1) \leq 2 - 2\varepsilon \\
 \frac{\gamma - 1}{\gamma} &\leq \frac{\gamma - 1}{\gamma} \frac{\gamma^t}{\gamma^t - 1} \leq 1.
 \end{aligned}$$

With these bounds, I can also bound the whole term in brackets in (S.2):

$$\begin{aligned}
 \frac{\gamma - 1}{\gamma} 2\varepsilon &\leq \frac{\gamma - 1}{\gamma} \frac{\gamma^t}{\gamma^t - 1} [1 + \rho(0) - \rho(1)] \leq 2 - 2\varepsilon \\
 \frac{\gamma - 1}{\gamma} 2\varepsilon - 1 &\leq \frac{\gamma - 1}{\gamma} \frac{\gamma^t}{\gamma^t - 1} [1 + \rho(0) - \rho(1)] - 1 \leq 1 - 2\varepsilon \\
 \left| 1 - \frac{\gamma - 1}{\gamma} \frac{\gamma^t}{\gamma^t - 1} [1 + \rho(0) - \rho(1)] \right| &\leq 1 - \frac{\gamma - 1}{\gamma} 2\varepsilon.
 \end{aligned}$$

This leads to a simple bound over time:

$$\begin{aligned}
 |E[p_{t+n} | I_t] - \rho^*| &= \prod_{\tau=t}^{t+n-1} \left| 1 - \frac{\gamma - 1}{\gamma} \frac{\gamma^\tau}{\gamma^\tau - 1} [1 + \rho(0) - \rho(1)] \right| |E[p_t | I_t] - \rho^*| \\
 &\leq \left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon \right)^{n-1}.
 \end{aligned}$$

In particular,

$$\begin{aligned}
 |E[p_{t+n} | a_t = 1] - \rho^*| &\leq \left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon \right)^{n-1} \\
 |E[p_{t+n}] - \rho^*| &\leq \left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon \right)^{t+n-1}.
 \end{aligned}$$

¹Note that $\rho(0) > \varepsilon$ and $\rho(1) < 1 - \varepsilon$, so $1 + \rho(0) - \rho(1) \geq 1 + \varepsilon - (1 - \varepsilon) = 2\varepsilon$ and so $1 + \rho(0) - \rho(1) \neq 0$.

So finally,

$$\begin{aligned}
|E[p_{t+n} | I_t] - E[p_{t+n}]| &\leq |E[p_{t+n} | a_t = 1] - \rho^*| + |E[p_{t+n}] - \rho^*| \\
&\leq \left(1 - \frac{\gamma-1}{\gamma} 2\varepsilon\right)^{n-1} + \left(1 - \frac{\gamma-1}{\gamma} 2\varepsilon\right)^{t+n-1} \\
&\leq 2 \left(1 - \frac{\gamma-1}{\gamma} 2\varepsilon\right)^{n-1},
\end{aligned}$$

and turning this into probabilities yields

$$\begin{aligned}
|\mathbf{P}_\sigma(a_{t+n} = 1 | a_t = 1) - \mathbf{P}_\sigma(a_{t+n} = 1)| &= |\rho(0) + [\rho(1) - \rho(0)]E[p_{t+n} | a_t = 1] \\
&\quad - [\rho(0) + [\rho(1) - \rho(0)]E[p_{t+n}]]| \\
&= |[\rho(1) - \rho(0)][E[p_{t+n} | a_t = 1] - E[p_{t+n}]]| \\
&\leq 2|E[p_{t+n} | a_t = 1] - E[p_{t+n}]| \\
&\leq 4 \left(1 - \frac{\gamma-1}{\gamma} 2\varepsilon\right)^{n-1} \\
&\leq \frac{4}{1 - \frac{\gamma-1}{\gamma} 2\varepsilon} \left(1 - \frac{\gamma-1}{\gamma} 2\varepsilon\right)^n.
\end{aligned}$$

Next assume that $\gamma = 1$. Then

$$\mathbf{P}_\sigma(a_t = 1) = \rho(0) + [\rho(1) - \rho(0)] \frac{1}{t-1} \sum_{\tau=1}^{t-1} a_\tau.$$

Define now $p_t \equiv \frac{1}{t-1} \sum_{\tau=1}^{t-1} a_\tau$ for $t \geq 2$, which leads to

$$p_{t+1} = \frac{1}{t} \sum_{\tau=1}^t a_\tau = \frac{t-1}{t} \sum_{\tau=1}^{t-1} a_\tau + \frac{1}{t} a_t = \frac{t-1}{t} p_t + \frac{1}{t} a_t.$$

In expectation,

$$\begin{aligned}
E[p_{t+1} | I_t] &= \frac{t-1}{t} E[p_t | I] + \frac{1}{t} E[a_t | I] \\
&= \frac{t-1}{t} E[p_t | I] + \frac{1}{t} E[\rho(0) + [\rho(1) - \rho(0)]p_t | I] \\
&= \frac{1}{t} [t-1 + \rho(1) - \rho(0)] E[p_t | I] + \frac{1}{t} \rho(0),
\end{aligned}$$

so in this case,

$$\begin{aligned}
E[p_{t+1} | I_t] - \rho^* &= \frac{1}{t} [t - 1 + \rho(1) - \rho(0)] E[p_t | I] + \frac{1}{t} \rho(0) - \rho^* \\
&= \frac{1}{t} [\rho(0) - [1 + \rho(0) - \rho(1)] E[p_t | I]] + E[p_t | I] - \rho^* \\
&= \frac{1}{t} [1 + \rho(0) - \rho(1)] [\rho^* - E[p_t | I]] + E[p_t | I] - \rho^* \\
&= \left[1 - \frac{1}{t} [1 + \rho(0) - \rho(1)] \right] [E[p_t | I] - \rho^*].
\end{aligned}$$

Then

$$E[p_{t+n} | I_t] - \rho^* = [E[p_t | I] - \rho^*] \prod_{\tau=0}^{n-1} \left[1 - \frac{1}{t+\tau} [1 + \rho(0) - \rho(1)] \right].$$

I present without proof the following remark.

REMARK 1. Let $0 < a_n < 1$ for all n . Then $\prod_{n=0}^{\infty} a_n > 0 \Leftrightarrow \sum_{n=0}^{\infty} (1 - a_n) < \infty$.

Then it suffices to show that

$$\sum_{\tau=0}^n \frac{1}{t+\tau} [1 + \rho(0) - \rho(1)] = [1 + \rho(0) - \rho(1)] \sum_{\tau=0}^n \frac{1}{t+\tau} = \infty$$

and follow the same steps as in the case with $\gamma > 1$. □

S.6. PROOF OF LEMMA 13

I show Proposition 1 by proving that $X|\sigma^T - E[X|\sigma^T]$ converges to zero in L^2 norm. The variance $V(\sigma^\tau)$ as defined by (5) is bounded above by

$$V(\sigma^\tau) \leq \frac{1}{T} \left(1 + 4(1 - 2\varepsilon^M)^{-1} \frac{(1 - 2\varepsilon^M)^{\frac{1}{M}}}{1 - (1 - 2\varepsilon^M)^{\frac{1}{M}}} \right).$$

Note that $\lim_{T \rightarrow \infty} 4(1 - 2\varepsilon^{M(T)})^{-1} = 4$ and $\lim_{T \rightarrow \infty} (1 - 2\varepsilon^{M(T)})^{\frac{1}{M(T)}} = 1$. Then the bound converges to zero whenever $\lim_{T \rightarrow \infty} T[1 - (1 - 2\varepsilon^{M(T)})^{\frac{1}{M(T)}}] = \infty$. I need to show that for any $K < \infty$, there exists a $\tilde{T} < \infty$ such that $T[1 - (1 - 2\varepsilon^{M(T)})^{\frac{1}{M(T)}}] \geq K$ for all $T \geq \tilde{T}$. This simplifies to

$$\left(1 - \frac{K}{T} \right)^{M(T)} \geq 1 - 2\varepsilon^{M(T)} \quad \forall T \geq \tilde{T}.$$

Since $(1 - \frac{K}{T})^{M(T)} \geq 1 - \frac{KM}{T}$, it suffices to show that

$$1 - \frac{KM}{T} \geq 1 - 2\varepsilon^{M(T)} \quad \Leftrightarrow \quad \frac{\varepsilon^{M(T)}}{M} \geq \frac{K}{2} \frac{1}{T},$$

where $M(T)$ is $o(\log(T))$. Then, for any constant $c \geq 0$, there is T large enough such that $M(T) \leq c \log(T)$. Pick $c = (-2 \log(\varepsilon))^{-1}$. Note next that the function ε^x/x is decreasing. Then, for T large, $\frac{\varepsilon^{M(T)}}{M(T)} \geq \frac{\varepsilon^{(-2 \log(\varepsilon))^{-1} \log(T)}}{(-2 \log(\varepsilon))^{-1} \log(T)}$. As a result, it suffices to show that for T large enough,

$$\begin{aligned} \frac{\varepsilon^{[(-2 \log(\varepsilon))^{-1} \log(T)]}}{(-2 \log(\varepsilon))^{-1} \log(T)} &\geq \frac{K}{2} \frac{1}{T} \\ \varepsilon^{(-2 \log(\varepsilon))^{-1} \log(T)} &\geq \frac{K}{2} \frac{1}{T} (-2 \log(\varepsilon))^{-1} \log(T) \\ T^{(-2 \log(\varepsilon))^{-1} \log(\varepsilon)} &\geq \frac{1}{-4 \log(\varepsilon)} K \frac{\log(T)}{T} \\ T^{-\frac{1}{2}} &\geq \frac{1}{-4 \log(\varepsilon)} K \frac{\log(T)}{T} \\ \frac{T^{\frac{1}{2}}}{\log(T)} &\geq \frac{1}{-4 \log(\varepsilon)} K. \end{aligned}$$

The left hand side goes to infinity and the right hand side is constant. Then there always exists a T such that this holds. This shows the first part of Proposition 1.

Next, I focus on the second part of Proposition 1. Equation (6) in the paper now becomes

$$\Pr\left(|X|\sigma^T - X|\tilde{\sigma}^T| \geq \frac{n}{T}\right) \leq \left[(1 - 2\varepsilon^{M(T)})^{\frac{1}{M(T)}}\right]^n,$$

which holds for all n .

Let $n = \lceil (-2 \log(\varepsilon))^{-1} \log(T) T^{\frac{3}{4}} \rceil$. As $(1 - 2\varepsilon^M)^{\frac{1}{M}} \leq 1$, then

$$\begin{aligned} \Pr\left(|X|\sigma^T - X|\tilde{\sigma}^T| \geq \frac{n}{T}\right) &\leq \left[(1 - 2\varepsilon^{M(T)})^{\frac{1}{M(T)}}\right]^n \\ &\leq \left[(1 - 2\varepsilon^{M(T)})^{\frac{1}{M(T)}}\right]^{(-2 \log(\varepsilon))^{-1} \log(T) T^{\frac{3}{4}}} \\ &\leq \left(1 - 2\varepsilon^{(-2 \log(\varepsilon))^{-1} \log(T)}\right)^{\frac{(-2 \log(\varepsilon))^{-1} \log(T) T^{\frac{3}{4}}}{(-2 \log(\varepsilon))^{-1} \log(T)}} \\ &= (1 - 2T^{-\frac{1}{2}})^{T^{\frac{3}{4}}}, \end{aligned}$$

where I have used the fact that $M(T)$ is $o(\log(T))$, so $M(T) \leq (-2 \log(\varepsilon))^{-1} \log(T)$ for T large enough. Moreover, I also used the fact that $(1 - 2\varepsilon^M)^{\frac{1}{M}}$ is increasing in M .

I need to show that for all $b > 0$, there exists $\tilde{T}$, such that $\Pr(|X|\sigma^T - X|\tilde{\sigma}^T| \geq b) < b$ for all $T > \tilde{T}$. Then it suffices to show that $\lim_{T \rightarrow \infty} \frac{n}{T} = 0$ and $\lim_{T \rightarrow \infty} (1 - 2T^{-\frac{1}{2}})^{T^{\frac{3}{4}}} = 0$.

So first, note that

$$\frac{n}{T} \leq \frac{(-2\log(\varepsilon))^{-1} \log(T) T^{\frac{3}{4}} + 1}{T} = \frac{1}{(-2\log(\varepsilon))} \frac{\log(T)}{T^{\frac{1}{4}}} + \frac{1}{T} \rightarrow 0,$$

so $\lim_{T \rightarrow \infty} \frac{n}{T} = 0$.

Second, note that $\lim_{T \rightarrow \infty} (1 - 2T^{-\frac{1}{2}}) T^{\frac{3}{4}} = 0 \Leftrightarrow \lim_{T \rightarrow \infty} T^{\frac{3}{4}} \log(1 - 2T^{-\frac{1}{2}}) = -\infty$. So using l'Hôpital's rule,

$$\lim_{T \rightarrow \infty} \frac{\log(1 - 2T^{-\frac{1}{2}})}{T^{-\frac{3}{4}}} = \lim_{T \rightarrow \infty} \frac{\frac{1}{1 - 2T^{-\frac{1}{2}}} (-2) \left(-\frac{1}{2}\right) T^{-\frac{3}{2}}}{-\frac{3}{4} T^{-\frac{7}{4}}} = \lim_{T \rightarrow \infty} -\frac{4}{3} \frac{T^{\frac{1}{4}}}{1 - 2T^{-\frac{1}{2}}} = -\infty.$$

This finishes the proof of the second part of Proposition 1.

Lemma 10 also needs some adjustment to allow for M to grow with T . Equation (8) from the paper becomes

$$\begin{aligned} \pi_{\theta}^T - E_{\sigma^T}[X_{\theta}] &= \frac{1}{T} \left[\sum_{\tau=1}^{M(T)-1} \overbrace{\mathbf{P}_{\sigma^T}(a_{\tau}=1)}^{\leq 1} \left(\sum_{t=\tau}^{\tau+M(T)-1} \overbrace{t^{-1}-1}^{\leq 1} \right) \right. \\ &\quad \left. - \sum_{\tau=T-M(T)+1}^T \overbrace{\mathbf{P}_{\sigma^T}(a_{\tau}=1)}^{\leq 1} \underbrace{\left(1 - \frac{T-\tau}{M(T)} \right)}_{\leq 1} \right] \\ &\leq \frac{2M(T)}{T}. \end{aligned}$$

Since $M(T)$ is $o(\log(T))$, then, $\pi_{\theta}^T - E_{\sigma^T} \rightarrow 0$. This adapts Lemma 10 to the case with growing M . The rest of Proposition 2 does not change.

S.7. MANY STATES OF THE WORLD AND MANY ACTIONS

S.7.1 The model

States and Actions There are N_{θ} equally likely states of the world $\theta \in \Theta = \{1, 2, \dots, N_{\theta}\}$. Agents must choose between N_a possible actions $a \in \mathcal{A} = \{1, 2, \dots, N_a\}$. Let $X^a \equiv \frac{1}{T} \sum_{j \in \mathcal{I}} \mathbb{1}\{a_j = a\}$ denote the proportion of agents who choose action a , with realizations $x^a \in [0, 1]$. The vector $X = (X^1, X^2, \dots, X^{N_a})$ denotes the proportion of agents who choose each action. Agent i obtains utility $u(a_i, X, \theta) : \mathcal{A} \times [0, 1] \times \Theta \rightarrow \mathbb{R}$, where $u(a_i, X, \theta)$ is a continuous function in X .

Private Signals Conditional on the true state of the world, signals are i.i.d. across individuals and distributed according to F_{θ} . I assume that F_{θ} and $F_{\tilde{\theta}}$ are mutually absolutely continuous for any two $\theta, \tilde{\theta} \in \Theta$. Then no perfectly revealing signals occur with positive probability, and the likelihood ratio (Radon–Nikodym derivative) $l_{\tilde{\theta}, \theta}(s) \equiv \frac{dF_{\tilde{\theta}}}{dF_{\theta}}(s)$ exists.

I also define a likelihood ratio that indicates how likely one state is, relative to all other states:

$$l_\theta(s) = \left(\sum_{\tilde{\theta} \neq \theta} l_{\tilde{\theta}, \theta}(s) \right)^{-1}.$$

Let $G_\theta(l) \equiv \Pr(l_\theta(S) \leq l \mid \theta)$. I modify the assumption of signals being of unbounded strength as follows.

DEFINITION (Signal strength). Signal strength is *unbounded* if $0 < G_\theta(l) < 1$ for all likelihood ratios $l \in (0, \infty)$ and for all states $\theta \in \Theta$.

Sampling Strategies and Mistakes The sampling rule does not change. A strategy is now a function $\sigma_i : S \times \Xi \rightarrow [\varepsilon, 1 - (N_a - 1)\varepsilon]^{N_a}$ that specifies a probability vector $\sigma_i(s, \xi)$ for choosing each action given the information available. For example, $\sigma_i^a(s, \xi)$ indicates the probability of choosing action $a \in \mathcal{A}$, after receiving signal s and sample ξ .

Definition of Social Learning I modify the definition of NE to allow for many states and actions. I say that x_θ corresponds to a Nash equilibrium of the stage game (and denote it by $x_\theta \in \text{NE}^\theta$) whenever $u(a, x_\theta, \theta) > u(a^*, x_\theta, \theta)$ for some $a, a^* \in \mathcal{A} \Rightarrow x_\theta^{a^*} = 0$. Then $x \in \text{NE}$ whenever $x_\theta \in \text{NE}^\theta$ for all $\theta \in \Theta$.

S.7.2 Results

Existence and Convergence of Average Action The proofs of Lemma 1 and Proposition 1 extend directly to a context with many actions and many states. I need to adapt the notation. The random variable $X \mid \sigma$ is now a matrix. Each element $X_\theta^a \mid \sigma$ is a random variable that denotes the proportion of agents choosing action a in state θ . So the random variable $X \mid \sigma = (X_1 \mid \sigma, X_2 \mid \sigma, \dots, X_{N_\theta} \mid \sigma)$ has realizations $x = (x_1, x_2, \dots, x_{N_\theta})$, where each x_θ is itself a vector: $x_\theta = (x_\theta^1, x_\theta^2, \dots, x_\theta^{N_a})$.

Utility Convergence In what follows, I provide modified expressions for the expected utility, the utility of the expected average action, and the approximate utility of a deviation. These expressions apply to contexts with many actions and many states.

Agents' expected utility under symmetric profile σ^T is simply

$$u(\sigma^T) \equiv E_{\sigma^T}[u(a_i, X, \theta)] = \frac{1}{N_\theta} \sum_{\theta \in \Theta} E_{\sigma^T} \left[\sum_{a \in \mathcal{A}} X_\theta^a \cdot u(a, X_\theta, \theta) \right].$$

Define the *utility of the expected average action* $\bar{u}^T$ by

$$\bar{u}^T \equiv \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} E_{\sigma^T}[X_\theta^a] \cdot u(a, E_{\sigma^T}[X_\theta], \theta).$$

Define the *approximate utility of the deviation* $\tilde{u}^T$ by

$$\tilde{u}^T \equiv \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} \mathbf{P}_{\tilde{\sigma}^T}(a_i = a \mid \theta) \cdot u(a, E_{\sigma^T}[X_\theta], \theta).$$

The proofs of Lemmas 2 and 3, as well as Corollary 1, extend directly to a context with many actions and many states.

COROLLARY 2' (The approximate improvement). *Let the approximate improvement Δ^T be given now by*

$$\Delta^T \equiv \tilde{u}^T - \bar{u}^T = \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} [\mathbf{P}_{\sigma^T}(a_i = a \mid \theta) - E_{\sigma^T}[X_\theta^a]] \cdot u(a, E_{\sigma^T}[X_\theta], \theta).$$

The proof of Corollary 2' extends directly to a context with many actions and many states.

S.7.3 Alternative Strategy 1: Always follow a given action

I present next a version of Lemma 4 that applies to many actions and many states. Let action $a^* \in \mathcal{A}$ be weakly dominant if

$$u(a^*, x_\theta, \theta) \geq u(a, x_\theta, \theta) \quad \text{for all } a \in \mathcal{A} \text{ and for all } \theta \in \Theta.$$

Let action $a^* \in \mathcal{A}$ be strictly dominant if

$$u(a^*, x_\theta, \theta) > u(a, x_\theta, \theta) \quad \text{for all } a \in \mathcal{A} \text{ and for all } \theta \in \Theta.$$

LEMMA 4' (Dominance). *If action $a^* \in \mathcal{A}$ is strictly dominant, then $x_\theta^{a^*} = 1 - (N_a - 1)\varepsilon$ for all $\theta \in \Theta$. Assume instead that action $a^* \in \mathcal{A}$ is weakly dominant. If there exists state $\theta \in \Theta$ with $u(a^*, x_\theta, \theta) > u(\tilde{a}, x_\theta, \theta)$, then $x_\theta^{\tilde{a}} = \varepsilon$.*

PROOF. Consider the alternative strategy of always choosing action a^* . Because of mistakes, this means a^* is chosen with probability $1 - (N_a - 1)\varepsilon$. Then the improvement is

$$\begin{aligned} \Delta^T &= \frac{1}{N_\theta} \sum_{\theta \in \Theta} \left[[1 - (N_a - 1)\varepsilon - x_\theta^{a^*}] u(a^*, x_\theta, \theta) + \sum_{a \neq a^*} (\varepsilon - x_\theta^a) \cdot u(a, x_\theta, \theta) \right] \\ &= \frac{1}{N_\theta} \sum_{\theta \in \Theta} \left[[1 - (N_a - 1)\varepsilon - x_\theta^{a^*}] u(a^*, x_\theta, \theta) - \sum_{a \neq a^*} (x_\theta^a - \varepsilon) \cdot u(a, x_\theta, \theta) \right]. \end{aligned}$$

Note that $x_\theta^a - \varepsilon \geq 0$ for all a, θ . Then

$$\begin{aligned} &[1 - (N_a - 1)\varepsilon - x_\theta^{a^*}] u(a^*, x_\theta, \theta) - \sum_{a \neq a^*} (x_\theta^a - \varepsilon) \cdot u(a, x_\theta, \theta) \\ &\geq [1 - (N_a - 1)\varepsilon - x_\theta^{a^*}] u(a^*, x_\theta, \theta) - \sum_{a \neq a^*} (x_\theta^a - \varepsilon) \cdot u(a^*, x_\theta, \theta) \\ &= \left[[1 - (N_a - 1)\varepsilon - x_\theta^{a^*}] - \sum_{a \neq a^*} (x_\theta^a - \varepsilon) \right] \cdot u(a^*, x_\theta, \theta) \end{aligned}$$

$$\begin{aligned}
&= \underbrace{\left[1 - (N_a - 1)\varepsilon - \sum_{a \in \mathcal{A}} x_\theta^a + (N_a - 1)\varepsilon \right]}_{=0} \cdot u(a^*, x_\theta, \theta) \\
&= 0.
\end{aligned}$$

Recall that $\Delta^T \leq 0$ by Corollary 2. Moreover, $\Delta^T \geq 0$. Then $\Delta^T = 0$. Also, as each term in Δ^T is weakly positive, then all terms in Δ^T must be zero:

$$[1 - (N_a - 1)\varepsilon - x_\theta^{a^*}]u(a^*, x_\theta, \theta) - \sum_{a \neq a^*} (x_\theta^a - \varepsilon) \cdot u(a, x_\theta, \theta) = 0.$$

Assume next that for some action $\tilde{a} \in \mathcal{A}$ in some state $\theta \in \Theta$, $u(a^*, x_\theta, \theta) > u(\tilde{a}, x_\theta, \theta)$. Then

$$\begin{aligned}
0 &= [1 - (N_a - 1)\varepsilon - x_\theta^{a^*}]u(a^*, x_\theta, \theta) - \sum_{a \neq a^*} (x_\theta^a - \varepsilon) \cdot u(a, x_\theta, \theta) \\
&\geq \left[1 - (N_a - 1)\varepsilon - x_\theta^{a^*} - \sum_{a \neq a^*, a \neq \tilde{a}} (x_\theta^a - \varepsilon) \right] u(a^*, x_\theta, \theta) - (x_\theta^{\tilde{a}} - \varepsilon)u(\tilde{a}, x_\theta, \theta) \\
&= [1 - \varepsilon - (1 - x_\theta^{\tilde{a}})]u(a^*, x_\theta, \theta) - (x_\theta^{\tilde{a}} - \varepsilon)u(\tilde{a}, x_\theta, \theta) \\
&= (x_\theta^{\tilde{a}} - \varepsilon)u(a^*, x_\theta, \theta) - (x_\theta^{\tilde{a}} - \varepsilon)u(\tilde{a}, x_\theta, \theta) \\
&= (x_\theta^{\tilde{a}} - \varepsilon)[u(a^*, x_\theta, \theta) - u(\tilde{a}, x_\theta, \theta)].
\end{aligned}$$

To sum up,

$$(x_\theta^{\tilde{a}} - \varepsilon) \overbrace{[u(a^*, x_\theta, \theta) - u(\tilde{a}, x_\theta, \theta)]}^{>0} \leq 0.$$

So $x_\theta^{\tilde{a}} = \varepsilon$. Similarly, if $u(a^*, x_\theta, \theta) > u(a, x_\theta, \theta)$ for all $a \in \mathcal{A}$ and for all $\theta \in \Theta$, then $x_\theta^{a^*} = 1 - (N_a - 1)\varepsilon$. $\square$

S.7.4 Alternative Strategy 2: Improve upon a sampled agent

Consider a possible limit point $x = (x_1, x_2, \dots, x_{N_\theta})$. Assume that action $\tilde{a}$ is not optimal in state θ^* : $u(a^*, x_{\theta^*}, \theta^*) > u(\tilde{a}, x_{\theta^*}, \theta^*)$, but it is still played in the limit: $x_{\theta^*}^{\tilde{a}} > \varepsilon$. As in the case with two states, let $\tilde{\xi}$ denote the action of one individual selected at random from the sample. Consider an alternative simple strategy $\tilde{\sigma}$, which makes the agent choose the action

$$\begin{aligned}
&a_i(\tilde{\xi}, s) \\
&= \begin{cases} a^* & \text{if } \tilde{\xi} = \tilde{a} \text{ and} \\ & l_{\theta^*}(s) \geq k^T \equiv \frac{-\bar{u}}{u(a^*, E_{\sigma^T}[X_{\theta^*}], \theta^*) - u(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*)} \frac{1}{\mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a} | \theta = \theta^*)} \\ \tilde{\xi} & \text{otherwise.} \end{cases}
\end{aligned}$$

I provide next a version of Lemma 5 in the paper that applies to many actions and many states.

LEMMA 5' (Improvement principle). *Take any limit point $x \in L$ with $u(a^*, x_{\theta^*}, \theta^*) > u(\tilde{a}, x_{\theta^*}, \theta^*)$. Then*

$$\begin{aligned} & \tilde{\Delta}(\varepsilon) + \frac{1 - (N_a - 1)\varepsilon}{N_\theta} [x_{\theta^*}^{\tilde{a}} \cdot [u(a^*, x_{\theta^*}, \theta^*) - u(\tilde{a}, x_{\theta^*}, \theta^*)]] \\ & \times [[1 - G_{\theta^*}(\bar{k})] - \bar{k}[1 - \tilde{G}_{\theta^*}(\bar{k})]] \leq 0 \end{aligned} \quad (\text{S.3})$$

with

$$\begin{aligned} \bar{k} &= -\bar{u}[(u(a^*, x_{\theta^*}, \theta^*) - u(\tilde{a}, x_{\theta^*}, \theta^*))x_{\theta^*}^{\tilde{a}}]^{-1} \\ \tilde{\Delta}(\varepsilon) &= \frac{\varepsilon}{N_\theta} \left[\sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} [1 - (N_a - 1)x_\theta^a] u(a, x_\theta, \theta) \right]. \end{aligned}$$

See Section S.7.5 for the proof.

The term $[[1 - G_{\theta^*}(\bar{k})] - \bar{k}[1 - \tilde{G}_{\theta^*}(\bar{k})]] \geq 0$ in (S.3) decreases in $\bar{k}$ (as shown later in Proposition 3). Moreover, with signals of unbounded strength, this term is strictly positive. Then, whenever $x_\theta^{\tilde{a}} > 0$, there is potential for improvement. The existence of mistakes may present such an improvement. Note, however, that $\lim_{\varepsilon \rightarrow 0} \tilde{\Delta}(\varepsilon) = 0$. Then when mistakes are unlikely, the potential for improvement dominates in (S.3).

S.7.5 Proof of Lemma 5'

Let $\rho_\theta^T(a|\tilde{a}) \equiv \mathbf{P}_{\sigma^T}(a_i = a|\theta, \tilde{\xi} = \tilde{a})$. In general, the improvement is given by

$$\begin{aligned} \Delta^T &= \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} \left[\varepsilon + [1 - (N_a - 1)\varepsilon] \sum_{a' \in \mathcal{A}} \rho_\theta(a|a') \mathbf{P}_{\sigma^T}(\tilde{\xi} = a'|\theta) \right. \\ &\quad \left. - E_{\sigma^T}[X_\theta^a] \right] u(a, E_{\sigma^T}[X_\theta], \theta) \\ &= \left[\frac{\varepsilon}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} u(a, E_{\sigma^T}[X_\theta], \theta) \right] \\ &\quad + \frac{1 - (N_a - 1)\varepsilon}{N_\theta} \left[\sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} \sum_{a' \in \mathcal{A}} \rho_\theta(a|a') \mathbf{P}_{\sigma^T}(\tilde{\xi} = a'|\theta) u(a, E_{\sigma^T}[X_\theta], \theta) \right] \\ &\quad - \frac{1 - (N_a - 1)\varepsilon}{N_\theta} \left[\sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} E_{\sigma^T}[X_\theta^a] u(a, E_{\sigma^T}[X_\theta], \theta) \right] \\ &\quad - \frac{(N_a - 1)\varepsilon}{N_\theta} \left[\sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} E_{\sigma^T}[X_\theta^a] u(a, E_{\sigma^T}[X_\theta], \theta) \right]. \end{aligned}$$

Let

$$\begin{aligned}\tilde{\Delta}^T(\varepsilon) &\equiv \frac{\varepsilon}{N_\theta} \left[\sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} u(a, E_{\sigma^T}[X_\theta], \theta) - (N_a - 1) \left[\sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} E_{\sigma^T}[X_\theta^a] u(a, E_{\sigma^T}[X_\theta], \theta) \right] \right] \\ &= \frac{\varepsilon}{N_\theta} \left[\sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} [1 - (N_a - 1) E_{\sigma^T}[X_\theta^a]] u(a, E_{\sigma^T}[X_\theta], \theta) \right]\end{aligned}$$

and

$$J(\varepsilon) \equiv \frac{1 - (N_a - 1)\varepsilon}{N_\theta}.$$

Then

$$\begin{aligned}\Delta^T &= \tilde{\Delta}^T(\varepsilon) + J(\varepsilon) \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} \left[\sum_{a' \in \mathcal{A}} \rho_\theta(a|a') \mathbf{P}_{\sigma^T}(\tilde{\xi} = a'|\theta) - E_{\sigma^T}[X_\theta^a] \right] \\ &\quad \times u(a, E_{\sigma^T}[X_\theta], \theta).\end{aligned}\tag{S.4}$$

However,

$$\begin{aligned}&= \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} \left[\sum_{a' \in \mathcal{A}} \rho_\theta(a|a') \mathbf{P}_{\sigma^T}(\tilde{\xi} = a'|\theta) - E_{\sigma^T}[X_\theta^a] \right] u(a, E_{\sigma^T}[X_\theta], \theta) \\ &= \frac{1}{N_\theta} \left[\sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} \sum_{a' \in \mathcal{A}} \rho_\theta(a|a') \mathbf{P}_{\sigma^T}(\tilde{\xi} = a'|\theta) u(a, E_{\sigma^T}[X_\theta], \theta) \right] \\ &\quad - \frac{1}{N_\theta} \left[\sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} E_{\sigma^T}[X_\theta^a] u(a, E_{\sigma^T}[X_\theta], \theta) \right] \\ &= \frac{1}{N_\theta} \left[\sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} \sum_{a' \in \mathcal{A}} \rho_\theta(a|a') \mathbf{P}_{\sigma^T}(\tilde{\xi} = a'|\theta) u(a, E_{\sigma^T}[X_\theta], \theta) \right] \\ &\quad - \frac{1}{N_\theta} \left[\sum_{\theta \in \Theta} \sum_{a' \in \mathcal{A}} E_{\sigma^T}[X_\theta^{a'}] u(a', E_{\sigma^T}[X_\theta], \theta) \right] \\ &= \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a' \in \mathcal{A}} \left[\sum_{a \in \mathcal{A}} \rho_\theta(a|a') \mathbf{P}_{\sigma^T}(\tilde{\xi} = a'|\theta) u(a, E_{\sigma^T}[X_\theta], \theta) \right. \\ &\quad \left. - E_{\sigma^T}[X_\theta^{a'}] u(a', E_{\sigma^T}[X_\theta], \theta) \right].\end{aligned}$$

As a result, the improvement in (S.4) can be expressed as

$$\begin{aligned}\Delta^T &= \tilde{\Delta}^T(\varepsilon) + J(\varepsilon) \sum_{\theta \in \Theta} \sum_{a' \in \mathcal{A}} \left[\sum_{a \in \mathcal{A}} \rho_\theta(a|a') \mathbf{P}_{\sigma^T}(\tilde{\xi} = a'|\theta) u(a, E_{\sigma^T}[X_\theta], \theta) \right. \\ &\quad \left. - E_{\sigma^T}[X_\theta^{a'}] u(a', E_{\sigma^T}[X_\theta], \theta) \right].\end{aligned}$$

In particular, for the simple strategy $\tilde{\sigma}$,

$$\begin{aligned}\Delta^T &= \tilde{\Delta}^T(\varepsilon) + J(\varepsilon) \sum_{\theta \in \Theta} [\rho_\theta(a^*|\tilde{a}) \mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta) u(a^*, E_{\sigma^T}[X_\theta], \theta) \\ &\quad + [1 - \rho_\theta(a^*|\tilde{a})] \mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta) u(\tilde{a}, E_{\sigma^T}[X_\theta], \theta) - E_{\sigma^T}[X_\theta^{\tilde{a}}] u(\tilde{a}, E_{\sigma^T}[X_\theta], \theta)] \\ &= \tilde{\Delta}^T(\varepsilon) + J(\varepsilon) \sum_{\theta \in \Theta} [\rho_\theta(a^*|\tilde{a}) \mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta) [u(a^*, E_{\sigma^T}[X_\theta], \theta) - u(\tilde{a}, E_{\sigma^T}[X_\theta], \theta)] \\ &\quad + [\mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta) - E_{\sigma^T}[X_\theta^{\tilde{a}}]] u(\tilde{a}, E_{\sigma^T}[X_\theta], \theta)].\end{aligned}$$

Let

$$\tilde{\Delta}^T \equiv J(\varepsilon) \sum_{\theta \in \Theta} [\mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta) - E_{\sigma^T}[X_\theta^{\tilde{a}}]] u(\tilde{a}, E_{\sigma^T}[X_\theta], \theta).$$

Then

$$\begin{aligned}\Delta^T &= \tilde{\Delta}^T(\varepsilon) + \tilde{\Delta}^T \\ &\quad + J(\varepsilon) \sum_{\theta \in \Theta} [\rho_\theta(a^*|\tilde{a}) \mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta) [u(a^*, E_{\sigma^T}[X_\theta], \theta) - u(\tilde{a}, E_{\sigma^T}[X_\theta], \theta)]] \\ &= \tilde{\Delta}^T(\varepsilon) + \tilde{\Delta}^T \\ &\quad + J(\varepsilon) \left[\sum_{\theta \in \Theta, \theta \neq \theta^*} [\rho_\theta(a^*|\tilde{a}) \mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta) [u(a^*, E_{\sigma^T}[X_\theta], \theta) - u(\tilde{a}, E_{\sigma^T}[X_\theta], \theta)]] \right. \\ &\quad \left. + \rho_{\theta^*}(a^*|\tilde{a}) \mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta^*) [u(a^*, E_{\sigma^T}[X_{\theta^*}], \theta^*) - u(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*)] \right].\end{aligned}$$

Now let

$$-\bar{u} \equiv \min_{a \in \mathcal{A}, a' \in \mathcal{A}, \theta \in \Theta, x_\theta \in [0,1]^{N_\theta}} [u(a, x_\theta, \theta) - u(a', x_\theta, \theta)].$$

This minimum exists since there is a finite number of states and actions, and the utility functions are continuous in X . Then

$$[u(a^*, E_{\sigma^T}[X_{\theta^*}], \theta^*) - u(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*)] \geq -\bar{u}.$$

Then

$$\begin{aligned}\Delta^T &\geq \tilde{\Delta}^T(\varepsilon) + \tilde{\Delta}^T + J(\varepsilon) \mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta^*) [u(a^*, E_{\sigma^T}[X_{\theta^*}], \theta^*) - u(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*)] \\ &\quad \times \left[-\frac{\bar{u} \sum_{\theta \in \Theta, \theta \neq \theta^*} [\rho_\theta(a^*|\tilde{a}) \mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta)]}{\mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta^*) [u(a^*, E_{\sigma^T}[X_{\theta^*}], \theta^*) - u(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*)]} + \rho_{\theta^*}(a^*|\tilde{a}) \right]\end{aligned}$$

$$\begin{aligned}
&= \tilde{\Delta}^T(\varepsilon) + \tilde{\Delta}^T + J(\varepsilon) \mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta^*) [u(a^*, E_{\sigma^T}[X_{\theta^*}], \theta^*) - u(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*)] \\
&\quad \times \left[\rho_{\theta^*}(a^*|\tilde{a}) - k^T \sum_{\theta \in \Theta, \theta \neq \theta^*} [\rho_{\theta}(a^*|\tilde{a}) \mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta)] \right] \\
&\geq \tilde{\Delta}^T(\varepsilon) + \tilde{\Delta}^T + J(\varepsilon) \mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta^*) [u(a^*, E_{\sigma^T}[X_{\theta^*}], \theta^*) - u(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*)] \\
&\quad \times \left[\rho_{\theta^*}(a^*|\tilde{a}) - k^T \sum_{\theta \in \Theta, \theta \neq \theta^*} \rho_{\theta}(a^*|\tilde{a}) \right] \\
&= \tilde{\Delta}_*^T \equiv \tilde{\Delta}^T(\varepsilon) + \tilde{\Delta}^T + J(\varepsilon) \mathbf{P}_{\sigma^T}(\tilde{\xi} = \tilde{a}|\theta^*) [u(a^*, E_{\sigma^T}[X_{\theta^*}], \theta^*) - u(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*)] \\
&\quad \times [[1 - G_{\theta^*}(k^T)] - k^T [1 - \tilde{G}_{\theta^*}(k^T)]].
\end{aligned}$$

Note that $\lim_{T \rightarrow \infty} \tilde{\Delta}^T = 0$. Let $\tilde{\Delta}(\varepsilon) \equiv \lim_{T \rightarrow \infty} \tilde{\Delta}^T(\varepsilon)$. Finally, note that, as in the proof in the paper, $\lim_{T \rightarrow \infty} k^T = \bar{k}$. Then

$$\begin{aligned}
\lim_{T \rightarrow \infty} \Delta_*^T &= \tilde{\Delta}(\varepsilon) + \frac{1 - (N_a - 1)\varepsilon}{N_{\theta}} [x_{\theta^*}^{\tilde{a}} [u(a^*, x_{\theta^*}, \theta^*) - u(\tilde{a}, x_{\theta^*}, \theta^*)]] \\
&\quad \times [[1 - G_{\theta^*}(\bar{k})] - \bar{k} [1 - \tilde{G}_{\theta^*}(\bar{k})]].
\end{aligned}$$

S.7.6 Strategic learning

Lemmas 4' and 5' are the main building blocks to show how Proposition 2 also applies to a context with many states and many actions. I present this formally.

PROPOSITION 2' (Strategic learning). *Assume signals are of unbounded strength. Then there is strategic learning.*

The proof of Proposition 3' requires modifying Proposition 3 and Lemma 11 in the paper. With these results in hand, the proof of Proposition 2' is analogous to the proof of Proposition 2 in the main text. Lemma 11 extends directly to a context with many actions and many states. I present next a version of Proposition 3 in the paper that applies to many states of the world.

PROPOSITION 3'. *For all $l \in (\underline{l}, \bar{l})$, $G_{\theta}(l)$ satisfies*

$$l > \frac{G_{\theta}(l)}{\tilde{G}_{\theta}(l)} \quad \text{and} \quad l < \frac{1 - G_1(l)}{1 - G_0(l)}. \quad (\text{S.5})$$

Moreover, if $k' \geq k$, then

$$[1 - G_1(k)] - k[1 - G_0(k)] \geq [1 - G_1(k')] - k'[1 - G_0(k')]. \quad (\text{S.6})$$

PROOF. The proof follows that from Proposition 11 in [Monzón and Rapp \(2014\)](#), but here the likelihood ratio G_θ indicates how likely state θ is relative to all other states. Note first that

$$\begin{aligned} l_\theta(s)^{-1} &= \sum_{\tilde{\theta} \neq \theta} l_{\tilde{\theta}, \theta}(s) = \sum_{\tilde{\theta} \neq \theta} \frac{dF_{\tilde{\theta}}}{dF_\theta}(s) \\ dF_\theta(s) l_\theta(s)^{-1} &= \sum_{\tilde{\theta} \neq \theta} dF_{\tilde{\theta}}(s) \\ dF_\theta(s) &= l_\theta(s) \sum_{\tilde{\theta} \neq \theta} dF_{\tilde{\theta}}(s). \end{aligned}$$

Recall that $\tilde{G}_\theta(L) \equiv \sum_{\tilde{\theta} \neq \theta} \Pr(l_\theta(s) \leq L \mid \tilde{\theta})$:

$$\begin{aligned} G_\theta(L) &= \int_{\{S \in \mathcal{S} : l_\theta(s) \leq L\}} dF_\theta = \int_{\{S \in \mathcal{S} : l_\theta(s) \leq L\}} l_\theta(s) \sum_{\tilde{\theta} \neq \theta} dF_{\tilde{\theta}}(s) \\ &< \int_{\{S \in \mathcal{S} : l_\theta(s) \leq L\}} L \sum_{\tilde{\theta} \neq \theta} dF_{\tilde{\theta}}(s) = L \sum_{\tilde{\theta} \neq \theta} \int_{\{S \in \mathcal{S} : l_\theta(s) \leq L\}} dF_{\tilde{\theta}}(s) \\ &= L \tilde{G}_\theta(L). \end{aligned}$$

Similarly,

$$\begin{aligned} 1 - G_\theta(L) &= \int_{\{S \in \mathcal{S} : l_\theta(s) > L\}} dF_\theta = \int_{\{S \in \mathcal{S} : l_\theta(s) > L\}} l_\theta(s) \sum_{\tilde{\theta} \neq \theta} dF_{\tilde{\theta}}(s) \\ &> \int_{\{S \in \mathcal{S} : l_\theta(s) > L\}} L \sum_{\tilde{\theta} \neq \theta} dF_{\tilde{\theta}}(s) = L \sum_{\tilde{\theta} \neq \theta} \int_{\{S \in \mathcal{S} : l_\theta(s) > L\}} dF_{\tilde{\theta}}(s) \\ &= L[1 - \tilde{G}_\theta(L)]. \end{aligned}$$

This shows that (S.5) holds. I move next to the second part. Take $k' > k$:

$$\begin{aligned} [1 - G_\theta(k)] - [1 - G_\theta(k')] &= G_\theta(k') - G_\theta(k) = \int_{S \in \mathcal{S} : k \leq l_\theta(S) \leq k'} dF_\theta \\ &= \int_{S \in \mathcal{S} : k \leq l_\theta(S) \leq k'} l_\theta(S) \sum_{\tilde{\theta} \neq \theta} dF_{\tilde{\theta}} \\ &\geq k \int_{S \in \mathcal{S} : k \leq l_\theta(S) \leq k'} \sum_{\tilde{\theta} \neq \theta} dF_{\tilde{\theta}} = k[\tilde{G}_\theta(k') - \tilde{G}_\theta(k)] \\ &= k[1 - \tilde{G}_\theta(k)] - k[1 - \tilde{G}_\theta(k')] \\ &\geq k[1 - \tilde{G}_\theta(k)] - k'[1 - \tilde{G}_\theta(k')]. \end{aligned}$$

Then

$$\begin{aligned} [1 - G_\theta(k)] - [1 - G_\theta(k')] &\geq k[1 - \tilde{G}_\theta(k)] - k'[1 - \tilde{G}_\theta(k')] \\ [1 - G_\theta(k)] - k[1 - \tilde{G}_\theta(k)] &\geq [1 - G_\theta(k')] - k'[1 - \tilde{G}_\theta(k')]. \end{aligned}$$

This shows that (S.6) holds. □

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