

Supplement to “Trade clustering and power laws in financial markets”

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MAKOTO NIREI

Graduate School of Economics, University of Tokyo

JOHN STACHURSKI

Research School of Economics, Australian National University

TSUTOMU WATANABE

Graduate School of Economics, University of Tokyo

This technical appendix provides derivations omitted in the main paper.

Verification for the signal examples

Example 1: A linear distribution Consider a signal X that follows

$$f_n^H(x) = \frac{1}{2} + \epsilon_n x \quad \text{and} \quad f_n^L(x) = \frac{1}{2}, \quad -1 \leq x \leq 1,$$

where $\epsilon_n = n^{-\xi}/3$ and $0 < \xi < 1$. In this section, we show that this signal satisfies Assumptions 1, 2, and 3.

Clearly, the densities are strictly positive and continuously differentiable. The likelihood ratio satisfies MLRP because

$$\ell_n(x) = \frac{f_n^H(x)}{f_n^L(x)} = 1 + 2\epsilon_n x$$

is strictly increasing in x . Moreover, $\ell_n(x) \rightarrow 1$ as $n \rightarrow \infty$ uniformly in $x \in [-1, 1]$, satisfying Assumption 2. The cumulative distributions are

$$F_n^H(x) = \int_{-1}^x \frac{1}{2} + \epsilon_n z \, dz = \frac{x+1}{2} + \frac{x^2-1}{2} \epsilon_n$$
$$F_n^L(x) = \int_{-1}^x \frac{1}{2} \, dz = \frac{x+1}{2}.$$

Hence, $\lambda_n(x) = 1 + (x-1)\epsilon_n$. This implies $\lambda_n''(x) = 0$, satisfying Assumption 3.

Makoto Nirei: nirei@e.u-tokyo.ac.jp

John Stachurski: john.stachurski@anu.edu.au

Tsutomu Watanabe: watanabe@e.u-tokyo.ac.jp

Finally, we investigate Assumption 1. We note that

$$\lim_{x \rightarrow 1} \log\left(\frac{\Lambda_n(x)}{\lambda_n(x)}\right) = \log\left(\frac{f_n^H(1)}{f_n^L(1)}\right) = \log(1 + 2\epsilon_n) = O(\epsilon_n).$$

Thus, to show that the signal satisfies Assumption 1, it suffices to show that $\log(\Lambda_n(x)/\lambda_n(x))$ is decreasing in $x \in [-1, 1]$. We have

$$\begin{aligned} \frac{d}{dx} \log\left(\frac{\Lambda_n(x)}{\lambda_n(x)}\right) &= \frac{d}{dx} \left[\log\left(\frac{1}{F_n^H(x)} - 1\right) - \log\left(\frac{1}{F_n^L(x)} - 1\right) \right] \\ &= \frac{1}{1 - F_n^L(x)} \frac{f_n^L(x)}{F_n^L(x)} - \frac{1}{1 - F_n^H(x)} \frac{f_n^H(x)}{F_n^H(x)} \\ &= \frac{(1 - F_n^H(x))F_n^H(x)f_n^L(x) - (1 - F_n^L(x))F_n^L(x)f_n^H(x)}{(1 - F_n^L(x))F_n^L(x)(1 - F_n^H(x))F_n^H(x)}. \end{aligned}$$

The denominator is positive. We inspect the numerator to find it negative:

$$\begin{aligned} &\left(1 - \frac{x+1}{2} - \frac{x^2-1}{2}\epsilon_n\right)\left(\frac{x+1}{2} + \frac{x^2-1}{2}\epsilon_n\right)\left(\frac{1}{2}\right) - \left(1 - \frac{x+1}{2}\right)\left(\frac{x+1}{2}\right)\left(\frac{1}{2} + x\epsilon_n\right) \\ &= \frac{(1-x-(x^2-1)\epsilon_n)((x+1)+(x^2-1)\epsilon_n) - (1-x)(x+1)(1+2x\epsilon_n)}{8} \\ &= \frac{x+1}{8} [(1-x-(x^2-1)\epsilon_n)(1+(x-1)\epsilon_n) - (1-x)(1+2x\epsilon_n)] \\ &= \frac{(x+1)(1-x)}{8} [(1+(1+x)\epsilon_n)(1+(x-1)\epsilon_n) - (1+2x\epsilon_n)] \\ &= \frac{(x+1)(1-x)}{8} [(1+x)\epsilon_n(1+(x-1)\epsilon_n) - (x+1)\epsilon_n] \\ &= -\frac{(x+1)^2(1-x)^2\epsilon_n^2}{8} < 0. \end{aligned}$$

Hence, $\log(\Lambda_n(x)/\lambda_n(x))$ is bounded from below by $\log(1 + 2\epsilon_n)$. Thus, Assumption 1 is satisfied.

Example 2: An exponential signal Consider a signal X that follows an exponential distribution with

$$f^H(x) = \frac{\mu e^{-\mu x}}{1 - e^{-\mu}} \quad \text{and} \quad f_n^L(x) = \frac{(\mu + \epsilon_n)e^{-(\mu + \epsilon_n)x}}{1 - e^{-(\mu + \epsilon_n)}}, \quad 0 \leq x \leq 1,$$

where $\epsilon_n = \delta_\epsilon n^{-\xi}$ is a positive sequence, and $\delta_\epsilon > 0$, $\mu > 2$, and $\xi \in (0, 1)$ are constants. In this section, we show that this signal satisfies Assumptions 1, 2, and 3.

The signal has the monotone increasing likelihood ratio $\ell_n(x) = (\mu/(1 - e^{-\mu}))((1 - e^{-(\mu + \epsilon_n)})/(\mu + \epsilon_n))e^{\epsilon_n x}$. Thus, the signal satisfies all the properties assumed in Section 2.2. In particular, f_n^s is continuously differentiable and strictly positive over common bounded support $\mathcal{X}$ and satisfies MLRP ($\ell_n'(x) > 0$) for any $x \in \mathcal{X}$. Moreover, ℓ_n converges to 1 uniformly on $\mathcal{X}$ and, therefore, satisfies Assumption 2.

Next we show that the signal satisfies Assumption 1. We have

$$F^H(x) = \frac{1 - e^{-\mu x}}{1 - e^{-\mu}}, \quad 1 - F^H(x) = \frac{e^{-\mu x} - e^{-\mu}}{1 - e^{-\mu}}$$

$$F_n^L(x) = \frac{1 - e^{-(\mu + \epsilon_n)x}}{1 - e^{-(\mu + \epsilon_n)}}, \quad 1 - F_n^L(x) = \frac{e^{-(\mu + \epsilon_n)x} - e^{-(\mu + \epsilon_n)}}{1 - e^{-(\mu + \epsilon_n)}},$$

$\Lambda_n = (1 - F^H)/(1 - F_n^L)$, and $\lambda_n = F^H/F_n^L$. Let $\delta_n := \log(\Lambda_n/\lambda_n)$. Then

$$\begin{aligned} \delta_n(x, \epsilon_n) &= \log\left(\frac{e^{-\mu x} - e^{-\mu}}{e^{-(\mu + \epsilon_n)x} - e^{-(\mu + \epsilon_n)}} \frac{1 - e^{-(\mu + \epsilon_n)x}}{1 - e^{-\mu x}}\right) \\ &= \log\left(\frac{e^{(\mu + \epsilon_n)x} - 1}{e^{\mu x} - 1}\right) - \log\left(\frac{e^{(\mu + \epsilon_n)(x-1)} - 1}{e^{\mu(x-1)} - 1}\right). \end{aligned}$$

Note that δ_n is an analytic function of ϵ_n and converges to 0 as $\epsilon_n \rightarrow 0$ for any $x \in \mathcal{X}$. Thus, the first-order Taylor expansion of δ_n around $\epsilon_n = 0$ yields

$$\begin{aligned} \delta_n(x, \epsilon_n) &= \left(\frac{x e^{\mu x}}{e^{\mu x} - 1} - \frac{(x-1)e^{\mu(x-1)}}{e^{\mu(x-1)} - 1}\right)\epsilon_n + O(\epsilon_n^2) \\ &= (h(x) - h(x-1))\epsilon_n + O(\epsilon_n^2), \end{aligned} \tag{*}$$

where $h(x) := x/(1 - e^{-\mu x})$. We note that $h(x)$ is strictly increasing in x :

$$h'(x) = \frac{1 - e^{-\mu x} - \mu x e^{-\mu x}}{(1 - e^{-\mu x})^2} > 0.$$

The inequality holds since $1 - e^{-y} - ye^{-y} > 0$ for any $y \neq 0$ and also since $h'(0) = 1/2$ by l'Hôpital's rule. Hence, $h(x) - h(x-1)$ is bounded below by a positive number uniformly on $\mathcal{X}$.

The term $O(\epsilon_n^2)$ can be made arbitrarily small (say, a half of the lower bound of $(h(x) - h(x-1))\epsilon_n$) for large enough n . Therefore, applying $\epsilon_n = \delta_\epsilon n^{-\xi}$ to (*) above, we see that there exist constants $\delta > 0$ and n_1 such that $\delta_n(x) > \delta n^{-\xi}$ for any $x \in \mathcal{X}$ and for all $n > n_1$. This confirms that the signal satisfies Assumption 1.

Finally, we show that the signal satisfies Assumption 3. Let us write $\mu_L := \mu + \epsilon_n$. For this particular signal, we have

$$\begin{aligned} \lambda_n(x) &= \frac{1 - e^{-\mu_L}}{1 - e^{-\mu}} \frac{1 - e^{-\mu x}}{1 - e^{-\mu_L x}} \\ \lambda'_n(x) &= \frac{1 - e^{-\mu_L}}{1 - e^{-\mu}} \frac{\mu e^{-\mu x}(1 - e^{-\mu_L x}) - \mu_L e^{-\mu_L x}(1 - e^{-\mu x})}{(1 - e^{-\mu_L x})^2}. \end{aligned}$$

Thus, we have

$$\lambda_n''(x) = \frac{1 - e^{-\mu_L}}{1 - e^{-\mu}} \frac{\left[-\frac{\mu^2}{e^{\mu x} - 1} + \frac{\mu_L^2}{e^{\mu_L x} - 1} \right] (1 - e^{-\mu_L x}) - 2\mu_L e^{-\mu_L x} \left[\frac{\mu}{e^{\mu x} - 1} - \frac{\mu_L}{e^{\mu_L x} - 1} \right]}{(1 - e^{-\mu_L x})^2 (1 - e^{-\mu x})^{-1}}. \quad (**)$$

Now we have that

$$\frac{d}{d\mu} \left(\frac{\mu}{e^{\mu x} - 1} \right) = \frac{e^{\mu x} - 1 - \mu x e^{\mu x}}{(e^{\mu x} - 1)^2}$$

is negative for $\mu x > 0$, because $y - 1 < y \log y$ for any $y > 1$. Hence, the term

$$-2\mu_L e^{-\mu_L x} \left[\frac{\mu}{e^{\mu x} - 1} - \frac{\mu_L}{e^{\mu_L x} - 1} \right]$$

in (**) is negative since $\mu_L > \mu$. Also, we have

$$\frac{d}{d\mu} \left(\frac{\mu^2}{e^{\mu x} - 1} \right) = \frac{2\mu e^{\mu x} (1 - e^{-\mu x} - \mu x/2)}{(e^{\mu x} - 1)^2}.$$

Note that $1 - e^{-y} - y/2$ is strictly negative at $y = 2$ and decreasing in y for $y > 2$. Hence, for any fixed $\mu > 2$, there exists an $x_c < 1$ such that the above derivative is negative for any $x \in [x_c, 1]$. Thus, $[-\frac{\mu^2}{e^{\mu x} - 1} + \frac{\mu_L^2}{e^{\mu_L x} - 1}](1 - e^{-\mu_L x})$ in (**) is negative in $x \in [x_c, 1]$ for any n , since $\mu_L > \mu$. Hence, there exists an x_c such that, for every n , $\lambda_n''(x) \leq 0$ holds for any $x \in [x_c, 1]$. Thus, we verify that the signal satisfies Assumption 3.

Derivation of $\lambda_n'(x_a) = \ell_n'(x_a)/2$ and $\Lambda_n'(x_b) = \ell_n'(x_b)/2$ for (8) and (9)

Using (8), we obtain

$$\begin{aligned} \lim_{x \rightarrow x_a} \lambda_n'(x) &= f_n^L(x_a) \lim_{x \rightarrow x_a} \frac{\ell_n(x) - \lambda_n(x)}{F_n^L(x)} \\ &= f_n^L(x_a) \frac{\ell_n'(x_a) - \lambda_n'(x_a)}{f_n^L(x_a)} \\ &= \ell_n'(x_a) - \lambda_n'(x_a), \end{aligned}$$

which implies $\lambda_n'(x_a) = \ell_n'(x_a)/2$.

Similarly, using (9), we obtain

$$\begin{aligned} \lim_{x \rightarrow x_b} \Lambda_n'(x) &= f_n^L(x_b) \lim_{x \rightarrow x_b} \frac{\Lambda_n(x) - \ell_n(x)}{1 - F_n^L(x)} \\ &= f_n^L(x_b) \frac{\Lambda_n'(x_b) - \ell_n'(x_b)}{-f_n^L(x_b)} \\ &= -(\Lambda_n'(x_b) - \ell_n'(x_b)), \end{aligned}$$

which implies $\Lambda_n'(x_b) = \ell_n'(x_b)/2$.

Supplement to Proof of Lemma 2

In this section, we show that the probability of $\Gamma(t)/n$ in (15) exceeding $n^{-\nu_0}$ for some $\nu_0 > 0$ converges to 0 as $n \rightarrow \infty$.

From Lemma 1, $K_t \equiv \Gamma(t+1) - \Gamma(1)$ asymptotically follows a Poisson distribution with mean t . Combining with inequalities $\sqrt{2\pi}e^{-k}k^{k+0.5} \leq k! \leq e^{1-k}k^{k+0.5}$ for any integer k , we obtain

$$\begin{aligned}
 \Pr(K_t \geq k) &= \sum_{K_t=k}^{\infty} t^{K_t} e^{-t} / K_t! \\
 &= \sum_{s=0}^{\infty} t^{k+s} e^{-t} / (k+s)! \\
 &= t^k e^{-t} \sum_{s=0}^{\infty} \frac{t^s}{s!} \frac{s!}{(k+s)!} \\
 &\leq t^k e^{-t} \sum_{s=0}^{\infty} \frac{t^s}{s!} \frac{e^{1-s} s^{s+0.5}}{\sqrt{2\pi} e^{-(k+s)} (k+s)^{k+s+0.5}} \\
 &= t^k e^{-t} \sum_{s=0}^{\infty} \frac{t^s}{s!} \frac{e^{k+1}}{\sqrt{2\pi} (k+s)^k} \left(\frac{s}{k+s} \right)^{s+0.5} \\
 &\leq t^k e^{-t} \sum_{s=0}^{\infty} \frac{t^s}{s!} \frac{e^{k+1}}{\sqrt{2\pi} k^k} \\
 &= \frac{e}{\sqrt{2\pi}} \left(\frac{te}{k} \right)^k.
 \end{aligned}$$

Now we consider a region $t \in [0, T]$ and let $k = n^{1-\nu_0}$ for some $\nu_0 \in (0, 1)$. The upper bound of $\Pr(K_T \geq k)$ becomes $(e/\sqrt{2\pi})(n^{\nu_0-1} T e)^{n^{1-\nu_0}}$, which converges to 0 from above as $n \rightarrow \infty$. Also note that $\Gamma(t)$ is nondecreasing in t . Thus, the probability of events in which $\Gamma(t)$ exceeds $k = n^{1-\nu_0}$ declines to 0 as $n \rightarrow \infty$.

Derivation of (13)

This section derives the asymptotic expression (13) from (12) by applying Stirling's formula $m! \sim \sqrt{2\pi m}(m/e)^m$ as $m \rightarrow \infty$.

Substituting Stirling's formula into (12), we obtain

$$\begin{aligned}
 \frac{b_o}{m} \frac{e^{-\phi m} (\phi m)^{m-b_o}}{(m-b_o)!} &\sim \frac{b_o}{m} \frac{e^{-\phi m+m-b_o} (\phi m)^{m-b_o}}{\sqrt{2\pi(m-b_o)} (m-b_o)^{m-b_o}} \\
 &= \frac{b_o}{m\sqrt{2\pi(m-b_o)}} e^{-\phi m+m-b_o+(m-b_o)\log \phi} \left(1 - \frac{b_o}{m} \right)^{-m+b_o}
 \end{aligned}$$

$$\begin{aligned}
&\sim \frac{b_o(\phi e)^{-b_o}}{m\sqrt{2\pi(m-b_o)}} e^{-(\phi-1-\log \phi)m} e^{b_o} \\
&\sim \frac{b_o\phi^{-b_o}}{\sqrt{2\pi}} \frac{e^{-(\phi-1-\log \phi)m}}{m^{1.5}}.
\end{aligned}$$

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