

Supplement to “Optimal contracts with a risk-taking agent”

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DANIEL BARRON

Kellogg School of Management, Northwestern University

GEORGE GEORGIADIS

Kellogg School of Management, Northwestern University

JEROEN SWINKELS

Kellogg School of Management, Northwestern University

These proofs refer to results in the text and the Appendices.

D.4 Proof of existence for $\underline{u} = -\infty$

Suppose that $\underline{u} = -\infty$, which corresponds to the agent having no limited liability constraint. This section gives conditions under which a unique solution to (P) exists and satisfies certain properties. Say that $u(\cdot)$ is *regular* if $\underline{w} = -\infty$ and $\frac{u'(w)u''(w)}{[u''(w)]^2} < 3$ for all $w \in \mathbb{R}$. These conditions are quite mild; in particular, the second condition means that $u'(\cdot)$ is not excessively convex, in the sense that it has local concavity everywhere greater than -2 . See [Prékopa \(1973\)](#) and [Borell \(1975\)](#) for details.

PROPOSITION 10. *Suppose $\pi(y) \equiv y$, $u(\cdot)$ is strictly concave and regular, and $\underline{u} = -\infty$. Then for any $a \geq 0$, there exists a unique contract $v(\cdot)$ that implements a at maximum profit. Furthermore, there exists $\bar{u} < \infty$ independent of $\underline{u}$ such that $v(\bar{y}) < \bar{u}$ and $v(\underline{y}) > -\bar{u}$.*

PROOF. Given Lemma 6, it is enough to show that for some $\underline{u}$, $v_{\underline{u}}(\underline{y}) > \underline{u}$. Assume not, so that, in particular, for all $\underline{u}$, $v_{\underline{u}}(\underline{y}) = \underline{u}$. We show that this leads to a contradiction. We henceforth restrict attention to $\underline{u} \leq 0$. For $\underline{u}$ sufficiently negative, it cannot be the case that $v_{\underline{u}}$ is linear. In particular, if $v_{\underline{u}}$ is linear, then since $v_{\underline{u}}(\bar{y}) > u_0 + c(a)$, we have that

$$\int v_{\underline{u}}(x) f_a(x|a) dx = \int v'_{\underline{u}}(x) (-F_a(x|a)) dx \geq \frac{u_0 + c(a) - \underline{u}}{\bar{y} - \underline{y}},$$

which diverges in $\underline{u}$, contradicting that $v_{\underline{u}}$ must satisfy (IC-FOC) with equality. Hence, for each $\underline{u}$, we can take a point $x_{\underline{u}} \in C_{v_{\underline{u}}}$, and derive $\lambda_{\underline{u}}$ and $\mu_{\underline{u}}$ as in the proof of Proposition 3.

Daniel Barron: d-barron@kellogg.northwestern.edu

George Georgiadis: g-georgiadis@kellogg.northwestern.edu

Jeroen Swinkels: j-swinkels@kellogg.northwestern.edu

Let $z_{\underline{u}}(\cdot) = \rho(\lambda_{\underline{u}} + \mu_{\underline{u}}l(\cdot|a))$, where we follow the convention that $\rho(s) = -\infty$ for $s \leq 0$. The contract $v_{\underline{u}}$ will, in general, differ from $z_{\underline{u}}$, since $z_{\underline{u}}$ need be neither concave nor satisfy the limited liability constraint. Note that $n_{\underline{u}}(\cdot) = \rho^{-1}(v_{\underline{u}}(\cdot)) - (\lambda_{\underline{u}} + \mu_{\underline{u}}l(\cdot|a)) \stackrel{s}{=} v_{\underline{u}}(\cdot) - z_{\underline{u}}(\cdot)$.

STEP 1. There is $\bar{\mu} < \infty$ such that $\mu_{\underline{u}} \leq \bar{\mu}$ for all $\underline{u}$.

PROOF. Applying a small positive amount of $t_{x_{\underline{u}}, \bar{y}}$ adds cost at rate at most $\rho^{-1}(\bar{u}) \times \int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}})f(x|a) dx$, adds incentives at rate $\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}})f_a(x|a) dx$, and relaxes (IR). It follows that

$$\mu_{\underline{u}} \leq \rho^{-1}(\bar{u}) \frac{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}})f(x|a) dx}{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}})f_a(x|a) dx}.$$

But, as in the proof that $|Q(\mathbf{0})| > 0$,

$$\begin{aligned} \frac{\partial}{\partial x_{\underline{u}}} \frac{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}})f(x|a) dx}{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}})f_a(x|a) dx} &= -\frac{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}})f_a(x|a) dx}{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}})f(x|a) dx} + \frac{\int_{x_{\underline{u}}}^{\bar{y}} f_a(x|a) dx}{\int_{x_{\underline{u}}}^{\bar{y}} f(x|a) dx} \\ &\leq 0, \end{aligned}$$

and so we can take

$$\bar{\mu} = \rho^{-1}(\bar{u}) \frac{\int (x - \underline{y})f(x|a) dx}{\int (x - \underline{y})f_a(x|a) dx} < \infty. \quad \triangleleft$$

STEP 2. There is $\underline{\mu} > 0$ and $\underline{u}^* > -\infty$ such that $\mu_{\underline{u}} \geq \underline{\mu}$ for all $\underline{u} < \underline{u}^*$.

PROOF. Choose $-\infty < \underline{u}^* \leq 0$ such that

$$\rho^{-1}(\underline{u}^*) < \frac{1}{2}\rho^{-1}(u_0 + c(a)), \quad (15)$$

$$c'(a) < \frac{u_0 + c(a) - \underline{u}^*}{\bar{y} - \underline{y}}, \quad (16)$$

where such a $\underline{u}^*$ exists since by assumption $\lim_{w \rightarrow -\infty} \frac{1}{u'(w)} = 0$. Let

$$r \equiv \sup_{\tau \in [\frac{1}{2}\rho^{-1}(u_0 + c(a)), \infty)} \rho'(\tau).$$

Since $\rho(1/u'(w)) = u(w)$, we have that

$$\rho'\left(\frac{1}{u'(w)}\right) = \frac{(u')^3}{-u''}(w),$$

from which

$$\frac{\rho''\left(\frac{1}{u'(w)}\right)}{\rho'\left(\frac{1}{u'(w)}\right)} = u'(w) \left(\frac{u'''(w)u'(w)}{(u''(w))^2} - 3 \right). \quad (17)$$

Since u is regular, it follows that $\rho'' < 0$ and so $r < \infty$. Let $\bar{l}_x = \max_x l_x(x|a)$ and choose $\underline{\mu} > 0$ such that

$$\underline{\mu} < \frac{1}{2} \frac{\rho^{-1}(u_0 + c(a))}{l(\bar{y}|a) - l(\underline{y}|a)}, \quad (18)$$

$$\underline{\mu} < \frac{1}{r\bar{l}_x} \frac{u_0 + c(a)}{\bar{y} - \underline{y}}. \quad (19)$$

Assume that for some $\underline{u} < \underline{u}^*$, $\mu_{\underline{u}} < \underline{\mu}$. We show that this leads to a contradiction, establishing the result.

Using Corollary 2 (which depends only on the necessity part of the proof of Proposition 3, which is proved in Appendix B) and the fact that $\bar{y}$ is free, $n(\bar{y}) \leq 0$, and so $\lambda_{\underline{u}} + \mu_{\underline{u}}l(\bar{y}|a) \geq \rho^{-1}(v_{\underline{u}}(\bar{y})) \geq \rho^{-1}(u_0 + c(a))$. Thus,

$$\begin{aligned} \lambda_{\underline{u}} + \mu_{\underline{u}}l(\underline{y}|a) &= \lambda_{\underline{u}} + \mu_{\underline{u}}l(\bar{y}|a) - \mu_{\underline{u}}(l(\bar{y}|a) - l(\underline{y}|a)) \\ &\geq \rho^{-1}(u_0 + c(a)) - \frac{1}{2} \frac{\rho^{-1}(u_0 + c(a))}{l(\bar{y}|a) - l(\underline{y}|a)} (l(\bar{y}|a) - l(\underline{y}|a)) \\ &= \frac{1}{2} \rho^{-1}(u_0 + c(a)), \end{aligned} \quad (20)$$

where the inequality follows from $\mu_{\underline{u}} < \underline{\mu}$ and (18).

Since $\underline{u} < \underline{u}^*$, and by (15), $\rho^{-1}(v_{\underline{u}}(\underline{y})) = \rho^{-1}(\underline{u}) < \frac{1}{2} \rho^{-1}(u_0 + c(a))$. Thus, using (20), $n(\underline{y})$ is strictly positive and it follows by Corollary 2 that $v_{\underline{u}}$ begins with a linear segment, the slope of which (by concavity) is at least

$$\frac{u_0 + c(a) - \underline{u}}{\bar{y} - \underline{y}} \geq \frac{u_0 + c(a)}{\bar{y} - \underline{y}}.$$

But using (20) and the definition of r , we have that for all x ,

$$\begin{aligned} z'_{\underline{u}}(x) &= \rho'(\lambda_{\underline{u}} + \mu_{\underline{u}}l(x|a))\mu_{\underline{u}}l_x(x|a) \\ &\leq r\mu_{\underline{u}}\bar{l}_x \\ &< \frac{u_0 + c(a)}{\bar{y} - \underline{y}}, \end{aligned}$$

where the strict inequality follows from (19). Hence, the initial linear segment of $v_{\underline{u}}$ crosses $z_{\underline{u}}$ at most once (from below). This implies that the entire contract is, in fact, linear with slope at least $(u_0 + c(a) - \underline{u})/(\bar{y} - \underline{y})$. In particular, let x_H be the right end

of the linear segment. If x_H is at or before the crossing point, then $v_{\underline{u}}$ violates (2) and so cannot be optimal by part (i) of Proposition 3. If $x_H < \bar{y}$ is after the crossing, then we violate Corollary 2. It follows that $v_{\underline{u}}$ generates incentives at least

$$\frac{u_0 + c(a) - \underline{u}^*}{\bar{y} - \underline{y}} > c'(a)$$

using (16). But we have shown that (IC-FOC) binds at $v_{\underline{u}}$, leading to the desired contradiction. $\triangleleft$

STEP 3. There is $u_0 + c(a) > u_c > -\infty$ such that if $\underline{u} < \underline{u}^*$ and $\rho(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) < u_c$, then $z_{\underline{u}}(\cdot)$ is concave at x .

PROOF. Note first that ρ is trivially concave anywhere that it is equal to $-\infty$ and that, by assumption, $\lim_{s \rightarrow 0} \rho(s) = -\infty$. Hence, it is enough to prove concavity where $\rho(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a)))$ is finite. But it follows from (17) and the fact that u is regular that $\lim_{t \downarrow 0} \rho''(t)/\rho'(t) = -\infty$ and so $\rho''(t)/\rho'(t)$ is negative for t below some t' . Assume $\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a)) < t'$. Then

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \rho(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) &= \frac{\partial}{\partial x} (\rho'(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) \mu_{\underline{u}} l_x(x|a)) \\ &= \rho''(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) (\mu_{\underline{u}} l_x(x|a))^2 \\ &\quad + \rho'(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) \mu_{\underline{u}} l_{xx}(x|a) \\ &= \frac{\rho''}{\rho'} (\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) \mu_{\underline{u}} + \frac{l_{xx}}{l_x^2}(x|a) \\ &\leq \frac{\rho''}{\rho'} (\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) \underline{\mu} + \frac{l_{xx}}{l_x^2}(x|a). \end{aligned}$$

The second term is bounded by assumption. The first term diverges to $-\infty$ as $\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a)) \rightarrow 0$. Hence, since ρ is monotone and since $\lim_{w \rightarrow -\infty} u'(w) = \infty$, the result follows. $\triangleleft$

STEP 4. As in the derivation of r in Step 2, let $\hat{r}$ be such that for all $t \geq \rho^{-1}(u_c)$, $\rho'(t) \leq \hat{r}$. Let $-\infty < \hat{\underline{u}} \leq \underline{u}^*$ satisfy

$$\hat{s} \equiv \frac{u_0 + c(a) - \hat{\underline{u}}}{\bar{y} - \underline{y}} \geq \max\{c'(a), \bar{\mu} \bar{l}_x \hat{r}\}$$

and assume that $\underline{u} < \hat{\underline{u}}$. Then $z_{\underline{u}}(\underline{y}) \leq \underline{u}$.

PROOF. Assume that $z_{\underline{u}}(\underline{y}) > \underline{u}$. Then, since $v_{\underline{u}}(\underline{y}) = \underline{u}$, $v_{\underline{u}}$ begins with a linear segment of positive length of slope at least $\hat{s}$, and so by Proposition 3 and part (i) of Definition 2, crosses $z_{\underline{u}}$ from below and is strictly above $z_{\underline{u}}$ for an interval of positive length as well. Let $x_{\underline{u},c}$ be defined by $z_{\underline{u}}(x_{\underline{u},c}) = u_c$. If $v_{\underline{u}}$ has its initial crossing of $z_{\underline{u}}$ at or before $x_{\underline{u},c}$, then

since $z_{\underline{u}}$ is concave until $x_{\underline{u},c}$, $v_{\underline{u}}$ remains above $z_{\underline{u}}$ until $x_{\underline{u},c}$. But then, since for $x > x_{\underline{u},c}$, $\hat{s} \geq z'_{\underline{u}}$, $v_{\underline{u}}$ in fact never re-crosses $z_{\underline{u}}$. Alternatively, if the initial crossing of $z_{\underline{u}}$ by $v_{\underline{u}}$ is after $x_{\underline{u},c}$, then again, since $v_{\underline{u}}$ has slope greater than $z'_{\underline{u}}$ for $x > x_{\underline{u},c}$, $v_{\underline{u}}$ never re-crosses $z_{\underline{u}}$. In either case, by Corollary 2, $v_{\underline{u}}$ is thus linear on all of $[\underline{y}, \bar{y}]$, a contradiction. $\triangleleft$

STEP 5. Let $u_{y_0} = u_0 + c(a) - c'(a)(\bar{y} - y_0) > -\infty$. Then $v_{\underline{u}}(y_0) \geq u_{y_0}$.

PROOF. Since $v_{\underline{u}}(\bar{y}) \geq u_0 + c(a)$, it follows that everywhere on $[\underline{y}, y_0]$, $v_{\underline{u}}(\cdot)$ is below the line $L(\cdot)$ that goes through $(y_0, v_{\underline{u}}(y_0))$ and $(\bar{y}, u_0 + c(a))$, and everywhere on $(y_0, \bar{y}]$, $v_{\underline{u}}(\cdot)$ is above $L(\cdot)$. Hence, since $f_a < 0$ on $[\underline{y}, y_0]$ and $f_a > 0$ on $(y_0, \bar{y}]$,

$$\begin{aligned} c'(a) &= \int v_{\underline{u}}(x) f_a(x|a) dx \\ &\geq \int L(x) f_a(x|a) dx \\ &= \frac{u_0 + c(a) - v_{\underline{u}}(y_0)}{\bar{y} - y_0}. \end{aligned}$$

Rearranging yields the desired result. $\triangleleft$

STEP 6. Choose $\infty > u_s < \min\{u_{y_0}, u_c, \rho(-\bar{\mu}l(\underline{y}|a)), \hat{u}\}$ small enough that for all $t \leq u_s$,

$$\rho'\left(\frac{1}{u'(u^{-1}(t))}\right) \bar{\mu} l_x \geq \hat{s}, \quad (21)$$

where $l_x = \min_x l_x(x|a) > 0$. Since $\rho'(\tau)$ diverges to ∞ as $\tau \downarrow 0$, and since $1/u'(u^{-1}(t))$ goes to 0 as $t \downarrow -\infty$, such a u_s is guaranteed to exist.

STEP 7. Choose $\underline{u} < u_s$. Let $\bar{z}_{\underline{u}}(\cdot) = \rho(\bar{\lambda}_{\underline{u}} + \bar{\mu}l(\cdot|a))$, where $\bar{\lambda}_{\underline{u}}$ solves $\rho(\bar{\lambda}_{\underline{u}} + \bar{\mu}l(\underline{y}|a)) = \underline{u}$. By Step 4, $z_{\underline{u}}(\underline{y}) \leq \underline{u}$, and so, since $\mu_{\underline{u}} \leq \bar{\mu}$, $z_{\underline{u}}(\cdot) \leq \bar{z}_{\underline{u}}(\cdot)$. Let $x_{\underline{u},s}$ be defined by $\bar{z}_{\underline{u}}(x_{\underline{u},s}) = u_s$. Since $\bar{\lambda}_{\underline{u}} + \bar{\mu}l(\underline{y}|a) = 1/u'(u^{-1}(\underline{u})) > 0$, it follows that $\bar{\lambda}_{\underline{u}} + \bar{\mu}l(y_0|a) \geq -\bar{\mu}l(\underline{y}|a)$ and, hence,

$$\rho(\bar{\lambda}_{\underline{u}} + \bar{\mu}l(y_0|a)) > \rho(-\bar{\mu}l(\underline{y}|a)) > u_s,$$

where the last inequality is by definition of u_s in Step 6. Thus, $x_{\underline{u},s} < y_0$.

STEP 8. For all $x < x_{\underline{u},s}$, $v_{\underline{u}}(x) \leq \bar{z}_{\underline{u}}(x)$.

PROOF. Let $x_{\underline{u},c}$ be defined by $z_{\underline{u}}(x_{\underline{u},c}) = u_c$. By construction, $\bar{z}_{\underline{u}}(\cdot)$ is concave where $x \leq x_{\underline{u},c}$. Using (21), $\bar{z}'_{\underline{u}}(\cdot) > \hat{s}$ for $x < x_{\underline{u},s}$ and $\bar{z}'_{\underline{u}}(\cdot) < \hat{s}$ for $x \geq x_{\underline{u},c}$. Assume that for some $\tilde{x} < x_{\underline{u},s}$, $v_{\underline{u}}(\tilde{x}) > \bar{z}_{\underline{u}}(\tilde{x}) \geq z_{\underline{u}}(\tilde{x})$. By Corollary 2, $v_{\underline{u}}$ is linear at $\tilde{x}$. If $v'_{\underline{u}}(\tilde{x}) \leq \bar{z}'_{\underline{u}}(\tilde{x})$, then, since $\bar{z}_{\underline{u}}$ is concave on $[\underline{y}, x_{\underline{u},s}]$ and again using Corollary 2, $v_{\underline{u}}$ is also above $\bar{z}_{\underline{u}}$ and, hence, is linear, for all x in $[\underline{y}, \tilde{x}]$. But then

$$v_{\underline{u}}(\underline{y}) - \bar{z}_{\underline{u}}(\underline{y}) \geq v_{\underline{u}}(\tilde{x}) - \bar{z}_{\underline{u}}(\tilde{x}) > 0,$$

contradicting that $v_{\underline{u}}(\underline{y}) = \underline{u}$. Thus, $v'_{\underline{u}}(\tilde{x}) > \bar{z}'_{\underline{u}}(\tilde{x}) > \hat{s}$. But then $v_{\underline{u}}$ remains linear and, hence, strictly above the concave function $z_{\underline{u}}$ at least until $x_{\underline{u},c}$. For $x \geq x_{\underline{u},c}$, $\bar{z}'_{\underline{u}}(\tilde{x}) \leq \hat{s}$, and so as before v can never re-cross $\bar{z}_{\underline{u}}$, and so a fortiori can never re-cross $z_{\underline{u}}$. Hence, $v_{\underline{u}}$ is linear on $[\tilde{x}, \bar{y}]$, with slope at least $\hat{s}$. Let L be the line that agrees with $v_{\underline{u}}$ on $[\tilde{x}, \bar{y}]$. To the left of $\tilde{x}$, $v_{\underline{u}}$, being concave, lies below L . But $\tilde{x} < x_{\underline{u},s} < y_0$ and so, since $f_a(\cdot|a)$ is negative on $[\underline{y}, \tilde{x}]$,

$$\int v_{\underline{u}}(x) f_a(x|a) dx \geq \int L(x) f_a(x|a) dx \geq \hat{s} > c'(a),$$

again a contradiction. $\triangleleft$

STEP 9. We show that $\lim_{\underline{u} \rightarrow -\infty} \int v_{\underline{u}}(x) f_a(x|a) dx = \infty$. For $\underline{u}$ sufficiently negative, this provides the necessary contradiction to the original supposition that $v_{\underline{u}}(\underline{y}) = \underline{u}$ for all $\underline{u}$, proving the result.

PROOF. By Step 8, for $\underline{u}$ sufficiently negative, $v_{\underline{u}}(x) \leq \bar{z}_{\underline{u}}(x)$ for all $x \leq x_{\underline{u},s}$. Let $v_{\underline{u}}^T$ truncate $v_{\underline{u}}$ to never pay more than u_s . Since $\max(0, v_{\underline{u}}(x) - u_s)$ is an increasing function, $\int \max(0, v_{\underline{u}}(x) - u_s) f_a(x|a) dx \geq 0$ and, hence, $\int v_{\underline{u}}(x) f_a(x|a) dx \geq \int v_{\underline{u}}^T(x) f_a(x|a) dx$. Note also that since $v_{\underline{u}}(y_0) > u_s$, $v_{\underline{u}}^T(x) = u_s$ for all $x \geq y_0$. Let $\bar{z}_{\underline{u}}^T$ similarly truncate $\bar{z}_{\underline{u}}$ to pay u_s to the right of $x_{\underline{u},s}$. Then $\bar{z}_{\underline{u}}^T$ is everywhere at least as large as $v_{\underline{u}}^T$, but equal to $v_{\underline{u}}^T$ everywhere to right of y_0 . Hence, since f_a is negative to the left of y_0 , we have

$$\int v_{\underline{u}}(x) f_a(x|a) dx \geq \int v_{\underline{u}}^T(x) f_a(x|a) dx \geq \int \bar{z}_{\underline{u}}^T(x) f_a(x|a) dx.$$

To arrive at a contradiction, it would thus be enough to show that $\int \bar{z}_{\underline{u}}^T(x) f_a(x|a) dx$ diverges as $\underline{u} \rightarrow -\infty$. But by Moroni and Swinkels (2014, Lemma 4), under our regularity conditions, $\int \bar{z}_{\underline{u}}(x) f_a(x|a) dx$ does diverge as $\underline{u} \rightarrow -\infty$.

Let

$$u^{**} = \rho(1 + \bar{\mu}(l(\bar{y}|a) - l(\underline{y}|a))) < \infty.$$

Then, for all $\underline{u}$ sufficiently negative that $\frac{1}{u'(u^{-1}(\underline{u}))} \leq 1$, $\bar{z}_{\underline{u}}(\bar{y}) \leq u^{**}$. Hence,

$$\begin{aligned} \int \bar{z}_{\underline{u}}(x) f_a(x|a) dx - \int \bar{z}_{\underline{u}}^T(x) f_a(x|a) dx &= \int (\bar{z}_{\underline{u}}(x) - \bar{z}_{\underline{u}}^T(x)) f_a(x|a) dx \\ &\leq \int_{y_0}^{\bar{y}} (\bar{z}_{\underline{u}}(x) - \bar{z}_{\underline{u}}^T(x)) f_a(x|a) dx \\ &\leq (u^{**} - u_s) \int_{y_0}^{\bar{y}} f_a(x|a) dx \\ &< \infty, \end{aligned}$$

where the first inequality follows because $\bar{z}_{\underline{u}}(x) - \bar{z}_{\underline{u}}^T(x)$ is weakly positive, and the second inequality follows because it is bounded above by $u^{**} - u_s$. $\square$

D.5 Agent reports x

In this section, we allow the agent to send a contractible message $\tilde{x}$ after he observes x but before y is realized. Payments can therefore depend on both $\tilde{x}$ and y , which allows the principal to discipline the agent from engaging in risk-taking. Restricting attention to the case where both parties are risk-neutral, we show that a linear contract is optimal in this setting.

Since the principal does not benefit from risk-taking, it is without loss to restrict attention to mechanisms that punish the agent as much as possible whenever his report does not match the final output: $s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}$ for some upper semicontinuous function $s(\cdot)$. Then the principal's problem is

$$\begin{aligned} \max_{a, s(\cdot)} \quad & \mathbb{E}_{F(\cdot|a), G} [y - s(y)\mathbb{I}_{\{y=\tilde{x}\}} + M\mathbb{I}_{\{y\neq\tilde{x}\}}] \\ \text{subject to} \quad & a, G, \tilde{x} \in \arg \max_{\tilde{a}, \tilde{G} \in \mathcal{G}, \tilde{x}} \{ \mathbb{E}_{F(\cdot|\tilde{a}), \tilde{G}} [s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}] - c(\tilde{a}) \}, \\ & \mathbb{E}_{F(\cdot|a), G} [s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}] - c(a) \geq u_0, \\ & s(\cdot) \geq -M, \end{aligned}$$

where $\tilde{x}$ maps x to a report made to the principal.

Fix $s(\cdot)$, and consider the agent's choice of G_x and $\tilde{x}$ following any intermediate output $x > \underline{y}$. Define

$$\lambda_s(x) = \max \{ \lambda : \lambda(y - \underline{y}) - M = s(y) \text{ for some } y \geq x \}.$$

Intuitively, $\lambda_s(x)$ is the smallest slope such that $\lambda_s(x)(y - \underline{y}) - M \geq s(y)$ for all $y \geq x$. We show that following intermediate output $x > \underline{y}$, the agent optimally chooses G_x and $\tilde{x}$ so that his expected payoff is $\lambda_s(x)(x - \underline{y}) - M$.²⁵

LEMMA 7. *For any $s(\cdot)$ and $x \in \mathcal{Y}$, the principal's expected payment to the agent equals*

$$\sigma_s(x) \equiv \max_{G_x, \tilde{x}} \{ \mathbb{E}_{G_x} [s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}] \} = \begin{cases} s(\underline{y}) & \text{if } x = \underline{y}, \\ \lambda_s(x)(x - \underline{y}) - M & \text{if } x > \underline{y}. \end{cases} \quad (22)$$

PROOF. Fix $s(\cdot)$ and $x > \underline{y}$. First, we show that there exists some G_x and $\tilde{x}$ such that $\mathbb{E}_{G_x} [s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}] = \lambda_s(x)(x - \underline{y}) - M$. By definition of $\lambda_s(\cdot)$, there exists a $\hat{y} \geq x$ such that $\lambda_s(x)(\hat{y} - \underline{y}) - M = s(\hat{y})$. Let $\tilde{x} = \hat{y}$ and $G_x(y) = (1 - p_{\hat{y}}) + p_{\hat{y}}\mathbb{I}_{\{y \geq \hat{y}\}}$, where $p_{\hat{y}} = \frac{x - \underline{y}}{\hat{y} - \underline{y}}$; i.e., $y = \underline{y}$ with probability $1 - p_{\hat{y}}$ and $y = \hat{y}$ with probability $p_{\hat{y}}$. Then the agent's expected payoff is

$$\begin{aligned} p_{\hat{y}}s(\hat{y}) - (1 - p_{\hat{y}})M &= \frac{x - \underline{y}}{\hat{y} - \underline{y}}s(\hat{y}) - \frac{\hat{y} - x}{\hat{y} - \underline{y}}M \\ &= \frac{x - \underline{y}}{\hat{y} - \underline{y}}[\lambda_s(x)(\hat{y} - \underline{y}) - M] - \frac{\hat{y} - x}{\hat{y} - \underline{y}}M \\ &= \lambda_s(x)(x - \underline{y}) - M. \end{aligned}$$

²⁵If $x = \underline{y}$, then the agent is compelled to choose $G_{\underline{y}}(y) = 1$, so his expected payoff is equal to $s(\underline{y})$.

Next we show that the agent cannot earn more than $\lambda_s(x)(x - \underline{y}) - M$ following intermediate output x . For any report $\tilde{x}$, the agent earns more than $-M$ only if $y = \tilde{x}$, so his optimal distribution G_x maximizes the probability that $y = \tilde{x}$ subject to the constraint that $\mathbb{E}_{G_x}[y] = x$. This is accomplished by choosing $G_x(\cdot)$ such that $y = \tilde{x}$ with some probability $p_{\tilde{x}}$ and $y = \underline{y}$ with probability $1 - p_{\tilde{x}}$, where $p_{\tilde{x}}\tilde{x} + (1 - p_{\tilde{x}})\underline{y} = x$. It suffices to show that the agent's expected payoff under this distribution is maximized if $\tilde{x} = \hat{y}$.

Suppose that there exists some $\tilde{x} \neq \hat{y}$ such that $p_{\tilde{x}}s(\tilde{x}) - (1 - p_{\tilde{x}})M > p_{\hat{y}}s(\hat{y}) - (1 - p_{\hat{y}})M = \lambda_s(x)(x - \underline{y}) - M$. Then there must exist some $\tilde{\lambda} > \lambda_s(x)$ such that $\tilde{\lambda}(\tilde{x} - \underline{y}) - M = s(\tilde{x})$, which contradicts the definition of $\lambda_s(x)$. Therefore, for all x , the agent's expected payoff equals $\lambda_s(x)(x - \underline{y}) - M$. $\square$

To see this result, recall that the agent earns $-M$ whenever his report does not equal the realized output. Therefore, if he misreports $\tilde{x} \neq x$, then he chooses G_x to maximize the probability that $y = \tilde{x}$. In particular, it is optimal for G_x to put weight on only two points, $\tilde{x}$ and $\underline{y}$. Given this $\tilde{x}$, the agent's payoff can be written as $p_{\tilde{x}}s(\tilde{x}) - (1 - p_{\tilde{x}})M$, where $p_{\tilde{x}}\tilde{x} + (1 - p_{\tilde{x}})\underline{y} = x$. It can be shown that the agent's payoff can be rewritten as $\lambda(x - \underline{y}) - M$, where $\lambda \leq \lambda_s(x)$. There exists some report $\tilde{x}$ that sets $\lambda = \lambda_s(x)$, proving the result.

Using [Lemma 7](#), we can rewrite the principal's problem as

$$\begin{aligned} \max_{a, s(\cdot)} \quad & \mathbb{E}_{F(\cdot|a)}[x - \sigma_s(x)] \\ \text{subject to} \quad & a \in \arg \max_{\tilde{a}} \{ \mathbb{E}_{F(\cdot|\tilde{a})}[\sigma_s(x)] - c(\tilde{a}) \}, \\ & \mathbb{E}_{F(\cdot|a)}[\sigma_s(x)] - c(a) \geq u_0, \\ & s(\cdot) \geq -M, \end{aligned}$$

where, for any contract $s(\cdot)$, $\sigma_s(\cdot)$ is given by (22).

Recall the definition of $s_a^L(\cdot)$ from Section 4. We show that if $a \geq 0$ is such that (LL) holds with equality after $\underline{y}$ under $s_a^L(\cdot)$, then $s_a^L(\cdot)$ implements a at maximum profit in this setting. Consequently, if (LL) binds for the optimal $a \geq 0$, then a linear contract is optimal as in Proposition 2.

PROPOSITION 11. *Fix any effort $a \geq 0$. If $s_a^L(\underline{y}) = -M$, then $s_a^L(\cdot)$ implements a at maximum profit.*

PROOF. Note that $\lambda_s(\cdot)$ is decreasing for any $s(\cdot)$ and, moreover, is constant for all $x \in \mathcal{Y}$ if $s(\cdot)$ is affine. Let $\hat{s}(\cdot)$ implement a at maximum profit and suppose there exists $x_L < x_H$ such that $\lambda_{\hat{s}}(x_L) > \lambda_{\hat{s}}(x_H)$.

Define $s_L(y) = \beta(y - \underline{y}) - M$, where β is chosen such that $\mathbb{E}_{F(\cdot|a)}[s_L(y) - \lambda_{\hat{s}}(y) \times (y - \underline{y}) + M] = 0$. Such a β exists by the intermediate value theorem because $\lambda_{\hat{s}}(y) \geq 0$ is finite. Since $\lambda_{\hat{s}}(\cdot)$ is strictly decreasing over some interval, there exists some $y^* \in (\underline{y}, \bar{y})$ such that $\lambda_{\hat{s}}(y) \geq \beta$ if and only if $y \leq y^*$. Then $\beta - \lambda_{\hat{s}}(y)$ is first negative and then

positive, $\int [\beta - \lambda_{\hat{s}}(y)](y - \underline{y})f(y|a) dy = 0$ by construction, and $\frac{f_a(\cdot|a)}{f(\cdot|a)}$ is strictly increasing, so Beesack's inequality implies that

$$\int [\beta - \lambda_{\hat{s}}(y)](y - \underline{y})f_a(y|a) dy > 0.$$

Therefore, $s_L(\cdot)$ implements some effort level $a' > a$, which implies that $\beta > c'(a)$.

Observe that $s_a^L(y) < s_L(y)$ for all $y > \underline{y}$, because $s_a^L(y) = -M$ by assumption and $c'(a) < \beta$. Moreover, $s_a^L(\cdot)$ implements a and satisfies both the individual rationality and the limited liability constraints. Therefore, $s_a^L(\cdot)$ implements effort a at strictly higher profit than $\hat{s}(\cdot)$. So $\lambda_{\hat{s}}(\cdot)$ must be constant and $\sigma_{\hat{s}}(\underline{y}) = -M$, in which case $s_a^L(\cdot)$ is also optimal. $\square$

D.6 Comparative static of optimal contract with respect to $\underline{y}$

This appendix considers how a^* changes with the lower bound $\underline{y}$ on output. A decrease in $\underline{y}$ implies that the agent can take on more severe left-tail risk by gambling over worse outcomes. We prove that a lower $\underline{y}$ makes it costlier for the principal to induce any nonzero effort level. As $\underline{y}$ approaches $-\infty$, inducing any positive effort becomes arbitrarily expensive and so the agent exerts no effort in the optimal contract.

COROLLARY 3. *Consider a decreasing sequence $\{\underline{y}_k\}_{k=0}^{\infty}$ with $\lim_{k \rightarrow \infty} \underline{y}_k = -\infty$. For each $k \geq 0$, consider $\mathcal{Y} = [\underline{y}_k, \bar{y}]$ and some output distribution $F_k(\cdot|a)$ that satisfies our assumptions (i.e., has full support on $[\underline{y}_k, \bar{y}]$, satisfies $\mathbb{E}_{F_k(\cdot|a)}[x] = a$, etc.), and let a_k^* be the corresponding optimal effort. Then $\lim_{k \rightarrow \infty} a_k^* = 0$, and if $\pi(y) \equiv y$, then a_k^* is decreasing in k .*

Proposition 2 implies that the principal's expected payment from inducing $a^* \geq 0$ equals $E_{F(\cdot|a^*)}[\pi(y - c'(a^*)(y - \underline{y}) + w)]$. For small enough $\underline{y}$, $s_{a^*}^L(y) = -M$. But then implementing $a^* > 0$ becomes arbitrarily costly as $\underline{y} \rightarrow -\infty$, in which case the principal is better off not motivating the agent at all. If the principal is risk-neutral, then we can show that the principal's profit under $s_{a^*}^L(\cdot)$ is supermodular in a^* and $\underline{y}$, so that a^* is increasing in $\underline{y}$.

PROOF OF COROLLARY 3. Fix $\hat{a} > 0$. Define

$$y_1 \equiv \min_{a \in [\hat{a}, a^{\text{FB}}]} \left\{ a - \frac{c(a) + u_0 + M}{c'(a)} \right\}$$

and

$$y_2 \equiv \min_{a \in [\hat{a}, a^{\text{FB}}]} \left\{ \frac{u^{-1}(u_0) - (1 - c'(a)a) - M}{c'(\hat{a})} \right\},$$

and note that since $c'(a) \geq c'(\hat{a}) > 0$ for all $a \geq \hat{a}$, $y_{\min} \equiv \min\{0, y_1, y_2\} > -\infty$.

Let $\underline{y} < y_{\min}$ and suppose toward a contradiction that there exists a distribution $F(\cdot|a)$ on $[\underline{y}, \bar{y}]$ such that effort $a^* \geq \hat{a}$ is optimal under $F(\cdot|a)$. Note first that Proposition 2 implies that the principal's expected payoff equals

$$\mathbb{E}_{F(\cdot|a^*)}[\pi(y - s_{a^*}^L(y))] = \mathbb{E}_{F(\cdot|a^*)}[\pi(y - c'(a^*)(y - \underline{y}) + \min\{M, c'(a^*)(a^* - \underline{y}) - c(a^*) - u_0\})].$$

Since $\underline{y} < y_1$, $c'(a^*)(a^* - \underline{y}) - c(a^*) - u_0 > M$. Furthermore, the principal's payoff is bounded above by

$$\pi((1 - c'(a^*))a^* + c'(a^*)\underline{y} + M)$$

by Jensen's inequality. Since $\underline{y} < \min\{0, y_2\}$, $(1 - c'(a))a + c'(a)\underline{y} + M < u^{-1}(u_0)$ for any $a \in [\hat{a}, a^{\text{FB}}]$. But then $a^* \geq \hat{a}$ cannot be optimal because it is strictly dominated by $a^* = 0$ and $s(\cdot) \equiv u^{-1}(u_0)$, a contradiction. Hence, for $\underline{y} < y_{\min}$, any distribution $F(\cdot|a)$, and any optimal a^* , it must be that $a^* < \hat{a}$. Since $\hat{a} > 0$ is arbitrary, $\lim_{\underline{y} \rightarrow -\infty} a^* = 0$.

Suppose $\pi(y) \equiv y$. To prove that a^* is increasing in $\underline{y}$, it suffices to show that the principal's payoff from implementing a in an optimal contract, $\Pi(a, \underline{y}) = a - c'(a)(a - \underline{y}) + w$, is supermodular in a and $\underline{y}$.

Recall that $w = \min\{M, c'(a)(a - \underline{y}) - c(a) - u_0\}$ is a function of $(a, \underline{y})$. Therefore,

$$\frac{\partial \Pi}{\partial a} = 1 - c''(a)(a - \underline{y}) - c'(a) + \frac{\partial w}{\partial a}$$

and so

$$\frac{\partial^2 \Pi}{\partial y \partial a} = c''(a) + \frac{\partial^2 w}{\partial y \partial a}.$$

But $\frac{\partial^2 w}{\partial y \partial a} = 0$ if $M < c'(a)(a - \underline{y}) - c(a) - u_0$ and $\frac{\partial^2 w}{\partial y \partial a} = -c''(a)$ otherwise. In either case, $\frac{\partial^2 \Pi}{\partial y \partial a} \geq 0$ and so optimal effort a^* is increasing in $\underline{y}$, as desired. $\square$

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